Lecture Notes

1 Brownian motion

1.1 Definitions

1.2 Construction of the Brownian motion

1.3 Simulation of Brownian motion

1.4 Properties of the Brownian motion

Monday, April 13

1.5 Reflection Principle

Definition 1.1. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ be a filtered space. Let $\{X_t\}$ be a stochastic process adapted to $\{\mathcal{F}_t\}$. We say that $\{X_t\}$ is a Markov process if

$$\forall A \in \mathcal{F}, \forall h \geq 0, \forall t \geq 0, \quad \mathbb{P}(X_{t+h} \in A|\mathcal{F}_t) = \mathbb{P}(X_{t+h} \in A|X_t).$$

Notation:

$$\mathbb{P}(X_{t+h} \in A|X_t) = \mathbb{P}(X_{t+h} \in A|\sigma(X_t)) = \mathbb{E}[1_A(X_{t+h})|\sigma(X_t)].$$

Theorem 1.1. A Brownian motion is a Markov process (w.r.t the same filtration).

Sketch of proof. We want to prove that

$$\mathbb{P}(B_{t+h} \in A|\mathcal{F}_t) = \mathbb{P}(B_{t+h} \in A|B_t),$$

equivalently,

$$\mathbb{E}[1_A(B_{t+h})|\mathcal{F}_t] = \mathbb{E}[1_A(B_{t+h})|\sigma(B_t)].$$

Let $\Phi \colon \mathbb{R} \to \mathbb{R}$ be measurable, then

$$\mathbb{E}[\Phi(B_{t+h})|\mathcal{F}_t] = \mathbb{E}[\Phi(B_{t+h} - B_t + B_t)|\mathcal{F}_t] = \mathbb{E}[g(X, B_t)|\mathcal{F}_t],$$

where $X = B_{t+h} - B_t$, which is independent of $\mathcal{F}_t$, and $g(x, y) = \Phi(x + y)$.

Since X is independent of $\mathcal{F}_t$, and B_t is $\sigma(B_t)$-measurable, $\mathbb{E}[g(X, B_t)|\mathcal{F}_t] = \mathbb{E}[g(X, B_t)|\sigma(B_t)]$. To prove this, start with functions g of the form $g(x, y) = 1_C(x)1_D(y)$, and use the fact that they approximate any Borel function. □

Definition 1.2. A random variable T is an $\{\mathcal{F}_t\}$-stopping time if

$$\forall t \geq 0, \quad \{T \leq t\} \in \mathcal{F}_t.$$

Proposition 1.3.

1. Every deterministic time is a stopping time.
2. If $\{T_n\}$ is a sequence of stopping time, the $\sup_n T_n$ is a stopping time.

Proof. 1. Exercise.
2. Fix $t \geq 0$. Then,

$$\{\sup_n T_n \leq t\} = \cap_n \{T_n \leq t\} \in \mathcal{F}_t.$$

□

Remark 1.4. In general, $\inf_n T_n$ is not a stopping time. Indeed, recalling that if $m = \inf(A)$, then for all $\varepsilon > 0$, there exists $a \in A$, such that $m \geq a - \varepsilon$. In particular we have

$$\{\inf_n T_n \leq t\} = \cap_{\varepsilon>0} \cup_{n\geq 1} \{T_n \leq t + \varepsilon\} \in \cap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon} = \mathcal{F}_t^+.$$

Since in general $\mathcal{F}_t \neq \mathcal{F}_t^+$, it follows that $\inf_n T_n$ is not a stopping time.

Similarly, note that, when $\mathcal{F}_t = \sigma(B_s : s \leq t)$,
1. If F is a closed set, then $T = \inf\{t \geq 0 : B_t \in F\}$ is a stopping time.
2. If O is open, then $T = \inf\{t \geq 0 : B_t \in O\}$ is not a stopping time.

Definition 1.5. A filtration $\{F_t\}$ is right-continuous if for all $t \geq 0$, $\mathcal{F}_t = \mathcal{F}_t^+$.

Example 1.6. The canonical filtration for a Brownian motion $\{B_t\}$:
Define

$$\mathcal{F}_t = \sigma(B_s : s \leq t), \quad t \geq 0,$$

and

$$\widetilde{\mathcal{F}}_t = \mathcal{F}_t^+ = \cap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}, \quad t \geq 0.$$

Then $\{\widetilde{\mathcal{F}}_t\}$ is a right-continuous filtration, and $\{B_t\}$ is adapted to $\{\widetilde{\mathcal{F}}_t\}$.

Proposition 1.7.
1. If $\{T_n\}$ is a sequence of $\{\mathcal{F}_t^+\}$-stopping times, then $\inf_n T_n$ is an $\{\mathcal{F}_t^+\}$-stopping time.
2. If O is open, then $T = \inf\{t \geq 0 : B_t \in O\}$ is an $\{\widetilde{\mathcal{F}}_t\}$-stopping time.

Definition 1.8. For a stopping time T, define

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t^+, \forall t \geq 0\}.$$

Theorem 1.2. $\mathcal{F}_T$ is a σ-algebra.

Proof. Same proof as in the discrete case. □

Wednesday, April 15

Definition 1.9. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ be a filtered space. Let $\{X_t\}$ be a stochastic process adapted to $\{\mathcal{F}_t\}$. We say that $\{X_t\}$ is a strong Markov process if for all stopping time T finite almost surely,

$$\forall A \in \mathcal{F}, \forall h \geq 0, \quad \mathbb{P}(X_{T+h} \in A | \mathcal{F}_T) = \mathbb{P}(X_{T+h} \in A | X_T).$$

Theorem 1.3. The Brownian motion is a strong Markov process.

Sketch of Proof. Note that $\{B_{T+t} - B_T\}_{t\geq 0}$ is a standard Brownian motion independent of $\mathcal{F}_T$. $\square$

Theorem 1.4 (Reflection principle)**.** Let T be a stopping time and $\{B_t\}$ be a standard Brownian motion.

If $M = (x, y)$, then the reflection of M with respect to the line passing through $(0, a)$ and parallel to the x-axis is $M^* = (x, 2a - y)$ (draw a picture).

For $t \geq 0$, define

$$B_t^* = B_t 1_{t\leq T} + (2B_T - B_t)1_{t>T}.$$

Then, $\{B_t^*\}$ is a standard Brownian motion.

Definition 1.10. The process B_t^* defined in Theorem 1.4 is called reflected Brownian motion.

Corollary 1.11. Let $\{B_t\}$ be a Brownian motion. Consider, for $t \geq 0$,

$$M_t = \sup_{0\leq s\leq t} B_s.$$

Then, $M_t \sim |Z|$, where $Z \sim \mathcal{N}(0, t)$. This means that supremum of Brownian motion path has a χ distribution.

Proof. First note that $\mathbb{P}(M_t \geq 0) = 1$ because $B_0 = 0$ a.s.

Fix $a > 0$. Let us find $\mathbb{P}(M_t \geq a)$. Consider $\{B_t^*\}$ the reflected Brownian motion with respect to the stopping time $T_a = \inf\{t \geq 0 : B_t = a\}$. Note that
i) $\{B_t \geq a\} \subset \{M_t \geq a\}$.
ii) $\{M_t \geq a\} \cap \{B_t < a\} = \{B_t^* > a\}$.
The point i) is clear. The inclusion $\subset$ of the point ii) is clear from the picture (after reflection, $B_t < a$ if and only if $B_t^* > a$). For the other inclusion $\supset$, if $\{B_t^* > a\}$, then either $\{B_t > a\}$ either $\{B_t < a\}$. The case $\{B_t > a\}$ is impossible because $B_t > a$ implies that $T_a < t$. Necessarily, $\{B_t < a\}$. Since $\{B_t < a\}$ and $\{B_t^* > a\}$, we have $T_a \leq t$ and thus $M_t \geq a$.

Thus, from ii) and Theorem 1.4,

$$\mathbb{P}(M_t \geq a, B_t < a) = \mathbb{P}(B_t^* > a) = \mathbb{P}(B_t > a).$$

Hence,

$$\mathbb{P}(M_t \geq a) = \mathbb{P}(M_t \geq a, B_t < a) + \mathbb{P}(M_t \geq a, B_t \geq a) = 2\mathbb{P}(B_t \geq a) = \mathbb{P}(|B_t| \geq a).$$

$\square$

Friday, April 17

1.6 Differentiability of the paths of the Brownian motion

Theorem 1.5. With probability 1, the paths of the Brownian motion are nowhere differentiable. Formally, let $\{B_t\}$ be a Brownian motion, then

$$\mathbb{P}\Big(\{\omega \in \Omega : \exists t_0 \in [0, +\infty), t \mapsto B_t(\omega) \text{ is differentiable at } t_0\}\Big) = 0.$$

Proof. **Step 1: [Setup]**

Without loss of generality, let us prove the result on $[0,1]$. Denote

$$A = \{\omega \in \Omega : \exists t_0 \in [0,1], t \mapsto B_t(\omega) \text{ is differentiable at } t_0\}.$$

We want to prove that $\mathbb{P}(A) = 0$. For $n \geq 3$ and $k \in \{1, \dots, n-2\}$, define

$$M_{k,n} = \max\{|B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}|, |B_{\frac{k+1}{n}} - B_{\frac{k}{n}}|, |B_{\frac{k}{n}} - B_{\frac{k-1}{n}}|\},$$

and

$$M_n = \min(M_{1,n}, \dots, M_{n-2,n}).$$

Step 2: The goal of Step 2 is to prove that

$$\forall \omega \in A, \exists M \in \mathbb{N}, \exists n_0 \in N, \forall n \geq n_0, M_n(\omega) \leq \frac{5}{n}(1+M).$$

Let $\omega \in A$ (we drop the dependence on ω next). Then there exists $t_0 \in [0,1]$ such that $t \mapsto B_t$ is differentiable at t_0. By definition of differentiability, there exists $L \in \mathbb{R}$ and $\delta > 0$ such that for all $t \in [0,1] \setminus \{t_0\}$, if $|t - t_0| \leq \delta$, then $|B_t - B_{t_0} - L(t-t_0)| \leq |t - t_0|$ (taking $\varepsilon = 1$). Hence, by triangular inequality, for all t such that $|t - t_0| \leq \delta$,

$$|B_t - B_{t_0}| \leq (1 + |L|)|t - t_0|.$$

Now, note that there exists $n_0 \geq 1$ and $k \in \{1, \dots, n_0\}$, such that

$$t_0 \in \left[\frac{k-1}{n_0}, \frac{k}{n_0}\right] \text{ and } \left|\frac{k+2}{n_0} - \frac{k-1}{n_0}\right| = \frac{3}{n_0} \leq \delta.$$

Let $n \geq n_0$. Then there exists $k \in \{1, \dots, n\}$ such that the above holds. Hence,

$$|B_{\frac{k}{n}} - B_{\frac{k-1}{n}}| \leq |B_{\frac{k}{n}} - B_{t_0}| + |B_{t_0} - B_{\frac{k-1}{n}}| \leq (1+|L|)\left(|\frac{k}{n} - t_0| + |t_0 - \frac{k-1}{n}|\right) \leq \frac{2}{n}(1+|L|).$$

Similarly, we have

$$|B_{\frac{k+1}{n}} - B_{\frac{k}{n}}| \leq \frac{3}{n}(1+|L|) \quad \text{and} \quad |B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}| \leq \frac{5}{n}(1+|L|).$$

We thus proved that for all $n \geq n_0$, there exists $k \in \{1, \dots, n\}$ such that

$$M_{k,n} \leq \frac{5}{n}(1+|L|).$$

By definition of M_n, this tells us that for all $n \geq n_0$,

$$M_n \leq \frac{5}{n}(1+|L|).$$

Now, just take any integer M greater than $|L|$ to conclude that

$$\forall \omega \in A, \exists M \in \mathbb{N}, \exists n_0 \in N, \forall n \geq n_0, M_n(\omega) \leq \frac{5}{n}(1+M).$$

Equivalently,

$$A \subset \cup_{M \in \mathbb{N}} \cup_{n_0 \in \mathbb{N}} \cap_{n \geq n_0} \left\{M_n \leq \frac{5}{n}(1+M)\right\}.$$

Step 3: The goal of Step 3 is to prove that

$$\forall M \in \mathbb{N}, \lim_{n\to+\infty} \mathbb{P}(M_n \le \frac{5}{n}(1+M)) = 0.$$

Let $n \ge 3$ and $k \in \{1, \ldots, n-2\}$. Denote,

$$X_1 = |B_{\frac{k}{n}} - B_{\frac{k-1}{n}}|, \quad X_2 = |B_{\frac{k+1}{n}} - B_{\frac{k}{n}}|, \quad X_3 = |B_{\frac{k+2}{n}} - B_{\frac{k+1}{n}}|.$$

Since $\{B_t\}$ is a Brownian motion, X_1, X_2, X_3 are i.i.d. with same distribution as $|Z|$ where $Z \sim \mathcal{N}(0, \frac{1}{n})$. Thus, the CDF of $M_{k,n} = \max(X_1, X_2, X_3)$ is

$$F_{M_{k,n}}(x) = \mathbb{P}(M_{k,n} \le x) = \mathbb{P}(X_1 \le x)^3, \quad x \in \mathbb{R}.$$

Note that

$$\mathbb{P}(X_1 \le x) = \mathbb{P}(|Z| \le x\sqrt{n}),$$

where $Z \sim \mathcal{N}(0,1)$. Hence,

$$\mathbb{P}(X_1 \le x) = \frac{2}{\sqrt{2\pi}} \int_0^{x\sqrt{n}} e^{-\frac{t^2}{2}}\, dt \le \frac{2x\sqrt{n}}{\sqrt{2\pi}}.$$

We deduce that for all $M \in \mathbb{N}$,

$$\mathbb{P}(M_{k,n} \le \frac{5}{n}(1+M)) \le \left[\frac{10}{\sqrt{2\pi}}(1+M)\frac{1}{\sqrt{n}}\right]^3 = \frac{C}{n^{\frac{3}{2}}},$$

where $C = \left[\frac{10}{\sqrt{2\pi}}(1+M)\right]^3$. Hence, by union bound,

$$\mathbb{P}\Bigg(M_n \le \frac{5}{n}(1+M)\Bigg) = \mathbb{P}\Bigg(\cup_{k=1}^{n-2}\left\{M_{k,n} \le \frac{5}{n}(1+M)\right\}\Bigg) \le \sum_{k=1}^{n-2} \mathbb{P}\Bigg(M_{k,n} \le \frac{5}{n}(1+M)\Bigg) \le \frac{C}{\sqrt{n}}.$$

We conclude that

$$\forall M \in \mathbb{N}, \lim_{n\to+\infty} \mathbb{P}(M_n \le \frac{5}{n}(1+M)) = 0.$$

Step 4: [Conclusion]
From Step 2,

$$\mathbb{P}(A) \le \mathbb{P}\Bigg(\cup_{M\in\mathbb{N}} \cup_{n_0\in\mathbb{N}} \cap_{n\ge n_0} \left\{M_n \le \frac{5}{n}(1+M)\right\}\Bigg).$$

Denote $B_{n_0} = \cap_{n\ge n_0}\{M_n \le \frac{5}{n}(1+M)\}$, and note that $\{B_{n_0}\}$ is an increasing sequence of sets, hence, from Step 3,

$$\begin{aligned}
\mathbb{P}\Bigg(\cup_{n_0\in\mathbb{N}} \cap_{n\ge n_0}\left\{M_n \le \frac{5}{n}(1+M)\right\}\Bigg) &= \lim_{n_0\to+\infty} \mathbb{P}\Bigg(\cap_{n\ge n_0}\left\{M_n \le \frac{5}{n}(1+M)\right\}\Bigg) \\
&\le \lim_{n_0\to+\infty} \mathbb{P}\Bigg(\left\{M_{n_0} \le \frac{5}{n_0}(1+M)\right\}\Bigg) \\
&= 0.
\end{aligned}$$

We conclude that

$$\mathbb{P}(A) \le \sum_{M\in\mathbb{N}} \mathbb{P}\Bigg(\cup_{n_0\in\mathbb{N}} \cap_{n\ge n_0}\left\{M_n \le \frac{5}{n}(1+M)\right\}\Bigg) = 0.$$

$\square$