1 Brownian motion

1.1 Definitions

1.2 Construction of the Brownian motion

Monday, April 6

Step 3: [The limit is a Brownian motion on $[0,1]$]

- **Continuity:** By construction, for all $\omega \in \Omega$, for all $n \in \mathbb{N}$, $t \mapsto B_t^{(n)}(\omega)$ is continuous on $[0,1]$. Since, almost surely, $\{B_n\}$ converges uniformly on $[0,1]$ to B, we deduce that, almost surely, $t \mapsto B_t(\omega)$ is continuous.

- Since for all $n \in \mathbb{N}$, $B_0^{(n)} = 0$, we deduce that $B_0 = 0$.

- **Stationary increments:** Let $t, s \in \mathcal{D}$. Then, there exists $m \in \mathbb{N}$ such that $t, s \in \mathcal{D}_m$. Hence, $B_t^{(m)} - B_s^{(m)} \sim \mathcal{N}(0, t-s)$. By construction, for all $t \in \mathcal{D}_m$, for all $n \geq m$, $B_t^{(n)} = B_t^{(m)}$. Hence
$$B_t - B_s = \lim_{n \to +\infty} B_t^{(n)} - B_s^{(n)} = \lim_{n \to +\infty} B_t^{(m)} - B_s^{(m)} = B_t^{(m)} - B_s^{(m)},$$
where the limit is understood as "almost sure convergence". Since $B_t^{(m)} - B_s^{(m)} \sim \mathcal{N}(0, t-s)$, we have $B_t - B_s \sim \mathcal{N}(0, t-s)$. Now, assume that $t, s \in [0,1]$. By density of $\mathcal{D}$ in $[0,1]$, there exist sequences $\{t_k\}, \{s_k\} \in \mathcal{D}$ such that $t = \lim_k t_k$ and $s = \lim_k s_k$. Since, almost surely, $t \mapsto B_t$ is continuous, we have, almost surely, $B_t = \lim_k B_{t_k}$ and $B_s = \lim_k B_{s_k}$. Since, for all k, $B_{t_k} - B_{s_k} \sim \mathcal{N}(0, t_k - s_k)$, we can conclude that $B_t - B_s \sim \mathcal{N}(0, t-s)$ (use, for example, characteristic functions).

- **Independent increments:** Same argument.

1.3 Simulation of Brownian motion

Fix an integer $n \in \mathbb{N}$. Given times $0 = t_0 < t_1 < \cdots < t_n$, generate $Z_1, \ldots, Z_n$ i.i.d. $\mathcal{N}(0,1)$. Define

$$\begin{aligned}
B_0 &= 0, \\
B_{t_1} &= \sqrt{t_1} Z_1, \\
B_{t_2} &= B_{t_1} + \sqrt{t_2 - t_1} Z_2 = \sqrt{t_1} Z_1 + \sqrt{t_2 - t_1} Z_2, \\
&\vdots \\
B_{t_n} &= B_{t_{n-1}} + \sqrt{t_n - t_{n-1}} Z_n = \sum_{i=1}^{n} \sqrt{t_i - t_{i-1}} Z_i
\end{aligned}$$

Using this construction, $\{B_t\}$ is a Brownian motion at times $0 = t_0 < t_1 < \cdots < t_n$. Indeed, it starts at 0, and for all $l \leq m < n$,

$$B_{t_n} - B_{t_m} = \sum_{i=1}^{n} \sqrt{t_i - t_{i-1}} Z_i - \sum_{i=1}^{m} \sqrt{t_i - t_{i-1}} Z_i = \sum_{i=m+1}^{n} \sqrt{t_i - t_{i-1}} Z_i,$$

which is Gaussian $\mathcal{N}(0, t_n - t_m)$, and is independent of B_{t_l}.

Wednesday, April 8

1.4 Properties of the Brownian motion

Definition 1.1. $\{X_t\}_{t\geq 0}$ is a Gaussian process if for all $n \in \mathbb{N}$, for all $t_1 < \cdots < t_n$, the random vector $(X_{t_1}, \ldots, X_{t_n})$ is multivariate Gaussian.

Theorem 1.1. $(X_1, \ldots, X_n)$ is multivariate Gaussian $\iff$ every linear combination of the X_i's is Gaussian, that is, for all $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, $\lambda_1 X_1 + \cdots + \lambda_n X_n$ is Gaussian $\iff$

$$\exists\, \mu \in \mathbb{R}^n, \exists\, A \in \mathbb{R}^{n\times m}, (X_1, \ldots, X_n) = \mu + A(Z_1, \ldots, Z_n),$$

where $Z_1, \ldots, Z_n$ are i.i.d. $\mathcal{N}(0, 1)$.

Theorem 1.2. A Brownian motion is a Gaussian process.

Proof. Define

$$Z_j = \frac{B_{t_j} - B_{t_{j-1}}}{\sqrt{t_j - t_{j-1}}}, \quad j = 1, \ldots, n.$$

In particular, the Z_j's are i.i.d. standard Gaussian $\mathcal{N}(0, 1)$. Note that

$$\begin{pmatrix} B_{t_1} \\ \vdots \\ \vdots \\ B_{t_n} \end{pmatrix} = \begin{pmatrix} \sqrt{t_1} & 0 & \cdots & 0 \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ \sqrt{t_1} & \sqrt{t_2 - t_1} & \cdots & \sqrt{t_n - t_{n-1}} \end{pmatrix} \begin{pmatrix} Z_1 \\ \vdots \\ \vdots \\ Z_n \end{pmatrix}.$$

Hence $\{B_t\}$ is a Gaussian process. □

Definition 1.2. Let $\{\mathcal{F}_t\}$ be a filtration. The germ σ-algebra is

$$\mathcal{F}_s^+ = \cap_{t>s} \mathcal{F}_t.$$

Remark 1.3.

1. In general $\mathcal{F}_s^+ \neq \mathcal{F}_s$. Indeed, let X be a non-constant random variable. Define $X_t = tX$, $t \geq 0$, and $\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t)$. Note that for all $t > 0$, $\mathcal{F}_t = \sigma(X)$. However,

$$\{\emptyset, \Omega\} = \mathcal{F}_0 \neq \cap_{t>0} \mathcal{F}_t = \sigma(X).$$

2. $\mathcal{F}_s^+$ represents an infinitesimal additional information into the future.

Theorem 1.3 (Blumenthal 0-1 Law)**.** Let $\{B_t\}$ be a Brownian motion. If $A \in \mathcal{F}_0^+$, then $\mathbb{P}(A) = 0$ or 1.

Corollary 1.4. Let $\{B_t\}$ be a standard Brownian motion. Define

$$T_1 = \inf\{t > 0 : B_t > 0\}, \qquad T_2 = \inf\{t > 0 : B_t = 0\}, \qquad T_3 = \inf\{t > 0 : B_t < 0\}.$$

Then, $\mathbb{P}(T_1 = 0) = \mathbb{P}(T_2 = 0) = \mathbb{P}(T_3 = 0) = 1$.

Proof. One has

$$\{T_1 = 0\} = \cap_{n\geq 1} \cup_{\varepsilon\in(0,\frac{1}{n})} \{B_\varepsilon > 0\}.$$

Hence, $\{T_1 = 0\} \in \mathcal{F}_0^+$. Note that for all $t > 0$,

$$\{B_t > 0\} \subset \{T_1 \leq t\},$$

hence

$$\mathbb{P}(T_1 \leq t) \geq \mathbb{P}(B_t > 0) = \frac{1}{2}.$$

We deduce that

$$\mathbb{P}(T_1 = 0) = \mathbb{P}(\cap_n\{T_1 \leq \frac{1}{n}\}) = \lim_n \mathbb{P}(T_1 \leq \frac{1}{n}) \geq \frac{1}{2}.$$

Since $\{T_1 = 0\} \in \mathcal{F}_0^+$, by Blumenthal 0-1 law, we conclude that $\mathbb{P}(T_1 = 0) = 1$.

By symmetry, (that is, $\{-B_t\}$ is a Brownian motion), $\mathbb{P}(T_3 = 0) = 1$.

With probability 1, $t \mapsto B_t$ is continuous and satisfies $\mathbb{P}(T_1 = 0) = \mathbb{P}(T_3 = 0) = 1$, hence by the intermediate value theorem, $\mathbb{P}(T_2 = 0) = 1$. □

Remark 1.5. In particular, Corollary 1.4 tells us that with proba 1, for all $\varepsilon > 0$, B_t hits 0 infinitely many times in the interval $(0, \varepsilon)$.

Theorem 1.4 (Long term behavior of Brownian motion)**.** Let $\{B_t\}$ be a Brownian motion, then

$$\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} = +\infty \text{ and } \liminf_{t\to+\infty} \frac{B_t}{\sqrt{t}} = -\infty.$$

Proof. Fix $M > 0$.

$$\mathbb{P}(\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} > M) = \mathbb{P}(\limsup_{t\to 0^+} \sqrt{t}B_{\frac{1}{t}} > M) = \mathbb{P}(\cap_{t>0} \cup_{0\leq s\leq t} \{\sqrt{s}B_{\frac{1}{s}} > M\})$$

Fact: $\{sB_{\frac{1}{s}}\}$ is a Brownian motion (Time inversion — see later).
Fact: $\{\limsup f_t > M\} = \limsup\{f_t > M\}$.
Note that $\sqrt{s}B_{\frac{1}{s}} = \frac{X_s}{\sqrt{s}}$, where $X_s = sB_{\frac{1}{s}}$ being a Brownian motion. Hence,

$$\cap_{t>0} \cup_{0\leq s\leq t} \{\sqrt{s}B_{\frac{1}{s}} > M\} = \cap_{t>0} \cup_{0\leq s\leq t} \{X_s > M\sqrt{s}\} \in \mathcal{F}_0^+.$$

By Blumenthal 0-1 law,

$$\mathbb{P}(\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} > M) = 0 \text{ or } 1.$$

Now, note that

$$\mathbb{P}(\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} > M) \geq \mathbb{P}(\limsup_{n\to+\infty} \frac{B_n}{\sqrt{n}} > M) = \mathbb{P}(\cap_{n\geq 1} \cup_{k\geq n} \{\frac{B_k}{\sqrt{k}} > M\})$$

$$= \lim_{n\to+\infty} \mathbb{P}(\cup_{k\geq n}\{\frac{B_k}{\sqrt{k}} > M\}) \geq \lim_{n\to+\infty} \mathbb{P}(\{\frac{B_n}{\sqrt{n}} > M\}) = \lim_{n\to+\infty} \mathbb{P}(\{B_1 > M\}) = \mathbb{P}(B_1 > M) > 0.$$

We conclude that

$$\mathbb{P}(\limsup_{t\to+\infty} \frac{B_t}{\sqrt{t}} > M) = 1.$$

It follows that

$$\mathbb{P}(\limsup_{t\to+\infty}\frac{B_t}{\sqrt{t}}=+\infty)=\mathbb{P}(\cap_{M>0}\limsup_{t\to+\infty}\frac{B_t}{\sqrt{t}}>M)=\lim_{M\to+\infty}\mathbb{P}(\limsup_{t\to+\infty}\frac{B_t}{\sqrt{t}}>M)=1.$$

By symmetry ($\{-B_t\}$ is a Brownian motion), we deduce that

$$\mathbb{P}(\liminf_{t\to+\infty}\frac{B_t}{\sqrt{t}}=-\infty)=1.$$

□

Remark 1.6. In other words, a Brownian motion is recurrent (each value $a\in\mathbb{R}$ is visited infinitely many often).

Friday, April 10

Definition 1.7. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space. Recall that a filtration $\{F_t\}_{t\geq 0}$ is a familly of sigma-algebras such that for all $s\leq t$, $\mathcal{F}_s\subset\mathcal{F}_t\subset\mathcal{F}$.

A process $\{M_t\}_{t\geq 0}$ is a $\{F_t\}$ continuous-time martingale if

i) For all $t\geq 0$, M_t is $\mathcal{F}_t$-measurable.
ii) For all $t\geq 0$, $\mathbb{E}[|M_t|]<+\infty$.
iii) For all $s\leq t$, $\mathbb{E}[M_t|\mathcal{F}_s]=M_s$.

Proposition 1.8. A Brownian motion is a continuous-time martingale.

Proof.

$$\mathbb{E}[B_t|\mathcal{F}_s]=\mathbb{E}[B_t-B_s+B_s|\mathcal{F}_s]=\mathbb{E}[B_t-B_s|\mathcal{F}_s]+B_s=B_s,$$

because B_t-B_s is independent of $\mathcal{F}_s$ and has expectation 0. □

Theorem 1.5 (Law of Large Numbers for Brownian motion)**.** For a Brownian motion $\{B_t\}$, $\lim_{t\to+\infty}\frac{B_t}{t}=0$ almost surely.

Proof. **Step 1:** Note that $B_n=B_1-B_0+\cdots+B_n-B_{n-1}$, so we can write

$$B_n=\sum_{k=1}^{n}X_k,$$

where $X_k=B_k-B_{k-1}$. Note that $\{X_k\}$ is a sequence of i.i.d. $\mathcal{N}(0,1)$ random variable. Hence, by the strong LLN, $\frac{B_n}{n}\to\mathbb{E}[B_1]=0$ almost surely.

Step 2: We will prove that

$$\sum_{n\geq 0}\mathbb{P}(\sup_{t\in[n,n+1]}|B_t-B_n|>n^{\frac{2}{3}})<+\infty.$$

Fix $n\geq 0$. Define, for $m\geq 0$ and $k\in\{0,\dots,2^m\}$,

$$X_k=B_{n+\frac{k}{2^m}}-B_n.$$

Since $\{B_{n+t} - B_n\}_{t\geq 0}$ is a Brownian motion, it is a martingale. It follows that $\{X_k\}$ is a discrete time martingale. We can thus apply Doob's inequality and obtain

$$\mathbb{P}(\sup_{0\leq k\leq 2^m} |X_k| > n^{\frac{2}{3}}) \leq \frac{\mathbb{E}[|X_{2^m}|^2]}{n^{\frac{4}{3}}} = \frac{\mathbb{E}[|B_{n+1} - B_n|^2]}{n^{\frac{4}{3}}} = \frac{1}{n^{\frac{4}{3}}}.$$

Because $t \mapsto B_t$ is continuous, we have

$$\{\sup_{t\in[n,n+1]} |B_t - B_n| > n^{\frac{2}{3}}\} = \cup_{m\geq 1}\{\sup_{0\leq k\leq 2^m} |X_k| > n^{\frac{2}{3}}\}.$$

Hence,

$$\mathbb{P}(\sup_{t\in[n,n+1]} |B_t - B_n| > n^{\frac{2}{3}}) = \lim_{m\to+\infty} \mathbb{P}(\sup_{0\leq k\leq 2^m} |X_k| > n^{\frac{2}{3}}) \leq \frac{1}{n^{\frac{4}{3}}}.$$

Step 3: Define, for $n \geq 0$,

$$A_n = \{\sup_{t\in[n,n+1]} |B_t - B_n| > n^{\frac{2}{3}}\}.$$

Since $\sum \mathbb{P}(A_n) < +\infty$, by Borel-Cantelli we have $\mathbb{P}(\limsup A_n) = 0$. This means that, for all ω in a set of probability 1,

$$\exists n_0 \geq 1, \forall n \geq n_0, \forall t \in [n, n+1], \left|\frac{B_t(\omega)}{t}\right| \leq \frac{n}{t}\left(\left|\frac{B_t(\omega) - B_n(\omega)}{n}\right| + \left|\frac{B_n(\omega)}{n}\right|\right) \leq \frac{1}{n^{\frac{1}{3}}} + \left|\frac{B_n(\omega)}{n}\right|,$$

which goes to 0 as $n \to +\infty$. □

Corollary 1.9 (Time Inversion)**.** Let $\{B_t\}$ be a Brownian motion. The process $\{X_t\}_{t\geq 0}$ defined by $X_t = tB_{\frac{1}{t}}$ for $t > 0$ and $X_0 = 0$, is a Brownian motion, for the natural filtration $\widetilde{\mathcal{F}}_t = \sigma(X_s : s \leq t)$.

Proof. **Continuity at 0:** From Theorem 1.5, we have

$$\lim_{t\to 0^+} X_t = \lim_{t\to+\infty} X_{\frac{1}{t}} = \lim_{t\to+\infty} \frac{B_t}{t} = 0 = X_0.$$

Gaussian Increments: Note that, for $s \leq t$,

$$X_t - X_s = (t - s)B_{\frac{1}{t}} - s(B_{\frac{1}{s}} - B_{\frac{1}{t}}),$$

which is $\mathcal{N}(0, t - s)$.

Independent Increments: Since $(X_t - X_s, X_s)$ is a bivariate Gaussian, we can conclude independence because $\mathbb{E}[(X_t - X_s)X_s] = 0$. □