University of Florida **MAA 4102/5104**

Homework #1 – **Algebra of sets, Mathematical induction, Proof techniques**

Exercise 1. Suppose that $A = \{1, 3, 5\}$, $B = \{1, 2, 4\}$, and $C = \{1, 8\}$. Find the following:

1. $(A \cup B) \cap C$.
2. $A \cup (B \cap C)$.
3. $(A \setminus C) \cup B$.
4. $(A \cap B) \times C$.
5. $C \times C$.
6. $\emptyset \cap A$.

Exercise 2. If A, B, C are sets, prove that

1. $A \subset B$ if and only if $A \cap B = A$.
2. $A \cap B = A \setminus (A \setminus B)$.
3. $A \cap B$ and $A \setminus B$ are disjoint, and $A = (A \cap B) \cup (A \setminus B)$.

Exercise 3. If $A = [1, 3)$, $B = (1, 4)$, $C = (2, 5]$, $D = [3, 5]$, shade the region in $\mathbb{R}^2$ that represents

1. $(A \times B) \cap (C \times D)$.
2. $(A \cap C) \times (B \cap D)$.
3. $(A \times B) \cup (C \times D)$.
4. $(A \cup C) \times (B \cup D)$.

Exercise 4.

1. Prove that the sum of two odd integers is even.
2. Prove that the product of two odd integers is odd.

Exercise 5. Use mathematical induction to prove the given statements:

1. $n^2 + n$ is divisible by 2, for all $n \in \mathbb{N}$.
2. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.
3. $\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2$.

4. If $x \in (0,1)$ is a fixed real number, then $0 < x^n < 1$ for all $n \in \mathbb{N}$.
5. $2^n < n!$ for all natural numbers $n \geq 4$.
6. $\cos(n\pi) = (-1)^n$ for all $n \in \mathbb{N}$.
7. $n^5 - n$ is divisible by 5 for every $n \in \mathbb{N}$.
8. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$ for all $n \in \mathbb{N}$.

Exercise 6. If x is some fixed real number, for each $n \in \mathbb{N}$, find the sum $1 - x - x^2 - \cdots - x^n$.

Exercise 7.

1. Prove the Pascal identity: $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$, for all $1 \leq k \leq n$.
2. Prove that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$ for all $n \in \mathbb{N}$.

Exercise 8. Find the sum $\sum_{k=1}^{50} 2^k$.

Exercise 9. Negate the following statements:

1. There exists $p > 0$ such that for every x we have $f(x+p) = f(x)$.
2. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, t \in D$, if $|x-t| < \delta$ then $|f(x) - f(t)| < \varepsilon$.
3. For all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in D$, if $0 < |x-a| < \delta$ then $|f(x) - A| < \varepsilon$.

Exercise 10. Prove that if q^2 is divisible by 3, then so is q.

Exercise 11. Prove that if $x \neq 0$ is rational and y is irrational, then the product xy is irrational.

Exercise 12. Prove that the given numbers are irrational:

1. $\sqrt{3}$
2. $\sqrt{6}$
3. $2^{1/3}$
4. $\sqrt{2} + \sqrt{3}$

Exercise 13. Consider the statement P : the sum of two irrational numbers is irrational.

1. Give an example of a case in which P is true.
2. Prove or disprove P.