Exercise 1. Show that \((x^n)^{(n)} = n!\), for all \(n \in \mathbb{N}\).

Exercise 2. Give an example of a function that is three times differentiable but not three times continuously differentiable.

Exercise 3. Show that the Cauchy function
\[
f(x) = \begin{cases} 
  e^{-\frac{1}{x^2}} & \text{if } x > 0 \\
  0 & \text{if } x \leq 0
\end{cases}
\]
is infinitely differentiable. Verify that every Taylor polynomial of \(f\) centered about 0 is identically zero.

Exercise 4. Give an example of an infinitely differentiable function \(f\) on \((0, +\infty)\), such that \(\lim_{x \to +\infty} f(x) = 0\) but \(\lim_{x \to +\infty} f'(x) \neq 0\).

Exercise 5. Let \(f: \mathbb{R} \to \mathbb{R}\) be twice differentiable on \(\mathbb{R}\). Prove that if \(f''(x) = 0\), then there exist \(a, b \in \mathbb{R}\) such that \(f(x) = ax + b\).

Exercise 6. Write Taylor’s formula for \(f(x) = e^x\) centered about 0.

Exercise 7. Find the \(n\)-th Taylor polynomial for the given functions
1. \(f(x) = \frac{1}{1-x}\), centered about 0.
2. \(f(x) = \sin(x)\), centered about 0.
3. \(f(x) = \cos(x)\), centered about 0.
4. \(f(x) = \ln(x)\), centered about 1.

Exercise 8. Use Taylor’s theorem to prove that \(1 - \frac{x^2}{2} \leq \cos(x)\), for all \(x \in \mathbb{R}\).

Exercise 9. Let \(f: (a, b) \to \mathbb{R}\) be twice differentiable. Let \(c \in (a, b)\) be a critical point, that is \(f'(c) = 0\).
1. Prove that if \(f''(c) > 0\), then \(c\) is a local minimum of \(f\).
2. Prove that if \(f''(c) < 0\), then \(c\) is a local maximum of \(f\).
3. Show that if \(f''(c) = 0\), then \(c\) may or may not be a local extremum of \(f\).