University of Florida **MAA 4102/5104**

Homework #2 – **Ordered field and a real number system - Some properties of real numbers**

Exercise 1. Suppose that $(F, +, \cdot)$ is a field and $a, b, c \in F$. Prove that

1. $(-a)(-b) = ab$.
2. If $a + b = c + b$ then $a = c$.
3. $-a - b = -(a + b)$.
4. if $a \neq 0$, then $a^{-1} \neq 0$ and $(a^{-1})^{-1} = a$.

Exercise 2. Suppose that $(F, +, \cdot, <)$ is an ordered field. Prove that

1. If $a \in F$ and $a > 0$, then $-a < 0$.
2. $0 < 1$.
3. If $a, b \in F$ and $ab > 0$, then a and b are of the same sign.
4. If $a, b \in F$ and $ab < 0$, then a and b are of opposite sign.

Exercise 3. Prove that if $r \geq 1$ is a real number, then $r^2 \geq r$ and $\frac{1}{r^2} \leq \frac{1}{r}$.

Exercise 4. If $S \neq \emptyset$ is a subset of real numbers that is bounded below, prove that $\inf(S)$ exists.

Exercise 5. Prove that if a set A has a supremum, then $\sup(A)$ is unique.

Exercise 6. If possible, give an example of a nonempty bounded subset of $\mathbb{Q}$ that

1. has a least upper bound and a maximum in $\mathbb{Q}$.
2. has a least upper bound but no maximum in $\mathbb{Q}$.
3. does not have a least upper bound in $\mathbb{Q}$.

Exercise 7. Prove that $\mathbb{Q}$ is not a complete ordered field.

Exercise 8. Prove that $2^{1/3} + \sqrt{3}$ is an algebraic number.

Exercise 9. If x is rational, then prove that x is algebraic. Is the converse true? Explain.

Exercise 10.

1. Can you find two rational numbers a and b such that a^b is irrational?
2. Can you find two irrational numbers α and β such that α^β is rational?

Exercise 11. If $a, b \in \mathbb{R}$, and $a < b + \varepsilon$ for any $\varepsilon > 0$, prove that $a \leq b$.

Exercise 12. Suppose that $a, b \in \mathbb{R}$. Prove that

1. If $b > 0$, then $|a| < b$ if and only if $-b < a < b$.
2. If $b > 0$, then $|a| > b$ if and only if $a < -b$ or $a > b$.
3. $|a| = \sqrt{a^2}$.
4. $|ab| = |a||b|$.
5. $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$, provided that $b \neq 0$.

Exercise 13.

1. If $a, b, c \in \mathbb{R}$, prove that $|a - b| \leq |a - c| + |c - b|$.
2. If $a, b, c \in \mathbb{R}$ such that $a < b < c$, prove that $|a - b| + |b - c| = |a - c|$.

Exercise 14. If $|f(x)| \leq M$ for all $x \in [a, b]$, prove that $-2M \leq f(x_1) - f(x_2) \leq 2M$ for any $x_1, x_2 \in [a, b]$.

Exercise 15. If $a_1, \ldots, a_n \in \mathbb{R}$, with $n \in \mathbb{N}$, prove that $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$.