University of Florida **MAA 4102/5104**

Homework #3 – **Convergence of sequences, Finite limits, monotone sequences**

Exercise 1. Determine whether the given sequence $\{a_n\}$ converges or diverges. In each case, prove your conclusion.

1. $a_n = \frac{1}{2n-3}$
2. $a_n = \frac{n}{n^2-2}$
3. $a_n = (-1)^n$
4. $a_n = \sqrt{n+1} - \sqrt{n}$
5. $$a_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \frac{1}{n} & \text{if } n \text{ is even} \end{cases}$$

Exercise 2.

1. Consider the sequence $\{a_n\}$, where $a_n = \frac{1+2+3+\cdots+n}{n^2}$. Show that $\{a_n\}$ converges to $\frac{1}{2}$.
2. Consider the sequence $\{a_n\}$, where $a_n = \frac{1}{n^3} + \frac{2^2}{n^3} + \frac{3^2}{n^3} + \cdots + \frac{n^2}{n^3}$. What value does this sequence converge to? Explain.

Exercise 3. Prove that if $\{a_n\}$ converges to A, then $\{|a_n|\}$ converges to $|A|$. Is the converse true? Explain.

Exercise 4. If $\{a_n\}$ is a sequence of real numbers, and if $\lim_{n\to\infty} a_{2n} = A$ and $\lim_{n\to\infty} a_{2n+1} = A$, prove that $\{a_n\}$ converges to A.

Exercise 5. Suppose that $\{a_n\}$ and $\{b_n\}$ are two sequences with $\lim_{n\to\infty} b_n = 0$. If there exist constants A and k and a positive integer n^* such that $|a_n - A| \leq k|b_n|$ for all $n \geq n^*$, prove that $\{a_n\}$ converges to A.

Exercise 6. Give an example of a sequence that is bounded but not convergent.

Exercise 7. If $\{a_n\}$ converges to A and there exists n_1 such that $a_n > 0$ for all $n \geq n_1$, prove that $A \geq 0$.

Exercise 8. If $\{a_n\}$ and $\{b_n\}$ converge to A and B respectively, and if there exists n_1 such that $a_n < b_n$ for all $n \geq n_1$, does it necessarily imply that $A < B$? Explain.

Exercise 9. Prove that the limit of a convergent sequence is unique.

Exercise 10. If $\{a_n\}$ and $\{b_n\}$ are sequences such that $\{a_n\}$ converges to 0, is it true that the sequence $\{a_nb_n\}$ converges to 0? Explain.

Exercise 11. Give an example of a sequence that diverges to $+\infty$ but is not eventually increasing.

Exercise 12. Give an example of a convergent sequence that does not attain a maximum value.

Exercise 13. Prove that the following sequences are monotone, or eventually monotone.

1. $a_n = \frac{n}{2^n}$
2. $a_n = \frac{n^2}{2^n}$
3. $a_n = \frac{3^n}{1+3^{2n}}$
4. $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n n!}$

Exercise 14. Determine whether the following sequences are convergent by deciding on monotonicity and boundedness.

1. $a_n = \frac{n+1}{2n+1}$
2. $a_n = \frac{n^2}{2^n}$
3. $a_n = \sum_{k=1}^{n} \frac{1}{2^k}$
4. $a_n = \frac{(-1)^{n+1}}{n}$
5. $a_n = \frac{2^n}{n!}$

Exercise 15. Consider the sequence $\{a_n\}$, where

$$a_n = \sum_{k=1}^{n} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \cdots + \frac{1}{n^2}$$

Prove that $\{a_n\}$ converges to a limit A, where $1 \leq A \leq 2$. Is it true that $\frac{5}{4} \leq A \leq 2$? Explain.

Exercise 16. Let $A > 0$. Consider the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined recursively by

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right), \quad a_0 > 0 \text{ arbitrary.}$$

Determine the limit of a_n.