• You must master the 1st semester part of this course – MAP 6472.

Exercise 1.
Let $N: (\Omega, \mathcal{F}) \to (\mathbb{N}, \mathcal{P}(\mathbb{N}))$ be a random variable with values in $\mathbb{N}$, and let $\{X_n\}_{n \geq 1}: (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a sequence of random variables.
Show that $X_N$ and $\sum_{k=1}^{N} X_k$ are random variables.

Exercise 2.
Let $A, B \subset \Omega$. Recall that $\sigma(\{A\})$ is the smallest $\sigma$-algebra on $\Omega$ containing $A$.
1. Describe $\sigma(\{A\})$ and $\sigma(\{A\}) \cup \sigma(\{B\})$.
2. Is $\sigma(\{A\}) \cup \sigma(\{B\})$ a $\sigma$-algebra in general? What about $\sigma(\{A\}) \cap \sigma(\{B\})$?

Exercise 3.
Let $X, Y: (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be random variables such that $X \leq Y$ a.s.
1. Show that $\mathbb{E}[X] \leq \mathbb{E}[Y]$.
2. Show that if $\mathbb{E}[X] = \mathbb{E}[Y]$, then $X = Y$ a.s.

Exercise 4.
Let $X: (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. Show that
\[
\forall A \in \mathcal{F}, \quad \mathbb{E}[X 1_A] = 0 \implies X = 0 \text{ a.s.}
\]

Exercise 5.
Let $\{X_n\}_{n \geq 1}$ be a decreasing sequence of non-negative random variables such that $X_1 \in L^1$ (that is $\mathbb{E}[|X_1|] < +\infty$). Show that $\{X_n\}$ converges to $X$ a.s. for some random variable $X$, and
\[
\lim_{n \to +\infty} \mathbb{E}[X_n] = \mathbb{E}[X].
\]
Is the statement still valid if $\mathbb{E}[|X_1|] = +\infty$?

Exercise 6.
Definition: Let $X: (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be a random variable. We denote by $\sigma(X)$ the smallest $\sigma$-algebra included in $\mathcal{P}(\Omega)$ such that $X$ is $(\Omega, \sigma(X))$-measurable.
1. Show that $\sigma(X) = X^{-1}(\mathcal{B}(\mathbb{R}))$.
2. $Y$ is $\sigma(X)$-measurable if and only if there exists a measurable function $g: (\mathbb{R}, \mathcal{B}(\mathbb{R})) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ such that $Y = g(X)$.
3. Show that $\sigma(X, Y) = \sigma(\sigma(X) \cup \sigma(Y)) = \sigma(X^{-1}(\mathcal{B}(\mathbb{R})) \cup Y^{-1}(\mathcal{B}(\mathbb{R})))$. 