Exercise 1. Let \( \{X_n\} \) be a sequence of i.i.d. random variables such that \( \mathbb{E}[X_1^2] < +\infty \). Take \( \mathcal{F}_n = \sigma\{X_1, \ldots, X_n\} \), and denote \( S_n = \sum_{k=1}^n X_k, S_0 = 0 \).

1. Define \( M_n = (S_n - n\mathbb{E}[X_1])^2 - n\text{Var}(X_1), n \geq 1 \). Prove that \( \{M_n\} \) is an \( \mathcal{F}_n \)-martingale.

2. Assume that \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2} \). For \( u \in \mathbb{R} \), define \( M_n^u = \cosh(u) - ne^{uS_n} \). Prove that \( \{M_n^u\} \) is an \( \mathcal{F}_n \)-martingale.

Exercise 2. Let \( \{X_n\} \) be an \( \mathcal{F}_n \)-sub-martingale (resp. super-martingale) such that all \( X_n \) have same distribution.

1. Prove that \( \{X_n\} \) is an \( \mathcal{F}_n \)-martingale.

2. Deduce that for all \( a \in \mathbb{R} \), \( X_n \wedge a \) and \( X_n \vee a \) are martingales.

Exercise 3. (Wald’s identity)

Let \( \{X_n\} \) be a sequence of i.i.d. random variables in \( L^1 \). Take \( \mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}, n \geq 1 \), and denote \( S_n = \sum_{k=1}^n X_k, S_0 = 0 \).

1. Let \( \tau \) be an \( \mathcal{F}_n \)-integrable stopping time. Prove that \( S_{\tau} \) is integrable, and
   \[ \mathbb{E}[S_{\tau}] = \mathbb{E}[X_1|\mathbb{E}[\tau]]. \]

2. Prove that if \( \mathbb{E}[X_1] \neq 0 \) and if \( \tau \) is an \( \mathcal{F}_n \)-stopping time such that \( \sup_n |\mathbb{E}[S_{n\wedge \tau}]| < +\infty \), then \( \tau \) is integrable.

Exercise 4.

Let \( \{X_n\} \) be a sequence of i.i.d. random variables such that \( \mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2} \). Denote \( S_n = \sum_{k=1}^n X_k, S_0 = 0 \), and \( T = \inf\{n \geq 0 : S_n = 1\} \). Prove that \( \mathbb{E}[T] = +\infty \).

Hint: One may use the Wald identity.

Exercise 5. (Doob’s inequality)

Let \( \{X_n\} \) be a non-negative \( \{\mathcal{F}_n\} \)-martingale. Denote \( \tau = \inf\{n \geq 0 : X_n \geq c\} \), where \( c > 0 \).

1. Prove that \( \tau \) is an \( \{\mathcal{F}_n\} \)-stopping time.

2. Prove that for all \( j \leq n \), \( \mathbb{E}[X_{j\tau=j}] = \mathbb{E}[X_{n\tau=j}] \).

3. Deduce that \( \mathbb{E}[X_{\tau\tau\leq n}] = \mathbb{E}[X_{n\tau\leq n}], \) and that \( c\mathbb{P}(\tau \leq n) \leq \mathbb{E}[X_{n\tau\leq n}] \).

4. Prove that \( \mathbb{P}(\sup_{k\leq n} X_k \geq c) \leq \frac{\mathbb{E}|X_n|}{c} \).
Exercise 6.

Let \( \{X_n\}_{n \geq 0} \) be a sequence of random variables in \( L^1 \), adapted to a filtration \( \{F_n\}_{n \geq 0} \). Prove that \( \{X_n\}_{n \geq 0} \) is an \( \{F_n\}_{n \geq 0} \)-martingale if and only if for all \( \{F_n\}_{n \geq 0} \)-stopping time \( T \) bounded, one has \( E[X_T] = E[X_0] \).

**Hint:** One may consider for all \( n \in \mathbb{N} \) and \( B \in F_n \), \( T = n1_B + (n+1)1_B \).

Exercise 7. (The gambler)

In a favorable game, given \( p \in \left( \frac{1}{2}, 1 \right) \) and a sequence of i.i.d. random variables \( \{\varepsilon_n\} \) such that \( P(\varepsilon_n = 1) = p \), \( P(\varepsilon_n = -1) = 1 - p \), where \( \varepsilon_n = 1 \) if the gambler wins at the \( n \)-th game and \( \varepsilon_n = -1 \) if he loses. The initial wealth of the gambler is \( x_0 > 0 \). Let \( X_n \) be his wealth at time \( n \), and let \( C_n \) be the amount of money he bets at the \( (n+1) \)-th game. The gambler cannot ask for a loan.

The model is as follows: for each \( n \in \mathbb{N} \), the gambler chooses \( C_n \) measurable with respect to \( F_n = \sigma(\varepsilon_1, \ldots, \varepsilon_n) \), with \( 0 \leq C_n \leq X_n \) and \( X_{n+1} = X_n + C_n\varepsilon_{n+1} \).

1. (A risky strategy.) Here, we take \( C_n = X_n \) for all \( n \in \mathbb{N} \).
   (a) Prove that this strategy maximize \( E[X_n] \) for all \( n \).
   (b) Define \( T = \inf\{n \in \mathbb{N} : X_n = 0\} \). Determine the distribution of \( T \), and deduce that \( T < +\infty \) a.s.
   (c) Prove that \( X_n = 0 \) for all \( n \geq T \), and that \( \{X_n\} \) converges to \( 0 \) a.s.

2. (A cautious strategy.) Here, we take \( C_n = \gamma_nX_n \), with \( 0 < \gamma_n < 1 \).
   (a) For \( n \geq 1 \), define \( M_n = \log(X_n) - \sum_{k=0}^{n-1} (p\log(1 + \gamma_k) + (1 - p)\log(1 - \gamma_k)) \) and \( M_0 = \log(x_0) \). Prove that \( \{M_n\} \) is an \( \{F_n\} \)-martingale.
   (b) Prove that the choice \( \gamma_n = 2p - 1 \), for all \( n \), maximize \( E[\log(X_n)] \).
   (c) For this strategy, that is \( \gamma_n = 2p - 1 \), compute the increasing process \( < M >_n \) and study the limit in \( L^2 \) and almost sure of the sequence \( \{\frac{\log(X_n)}{n}\} \).