# A lower bound on the differential entropy for log-concave random variables with applications to rate-distortion theory


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***Abstract*—We derive a lower bound on the differential entropy for symmetric log-concave random variable $X$ in terms of the $p$-th absolute moment of $X$, which shows that entropy and $p$-th absolute moment of a symmetric log-concave random variable are comparable. We apply our bound to study the rate distortion function under distortion measure $|x - \hat{x}|^r$ for sources that follow a log-concave probability distribution. In particular, we establish that the difference between the rate distortion function and the Shannon lower bound is at most $\log(\sqrt{2}e) \approx 1.9$ bits, independently of $r$ and $d$. For mean-square error distortion, the difference is at most $\log\sqrt{\pi e} \approx 1.55$ bits, regardless of $d$. Our results generalize to the case of vector $X$. Our proof technique leverages tools from convex geometry.**



**Keywords.** Differential entropy, rate-distortion function, Shannon lower bound, log-concave distribution.


## I. Introduction

It is well known that the differential entropy among all zero-mean random variables with the same second moment is maximized by the Gaussian distribution:

$$h(X) \le \log(\sqrt{2\pi e \mathbb{E}\left[|X|^2\right]}). \tag{1}$$

More generally, the differential entropy under $p$-th moment constraint is upper bounded as (see e.g. [1, Appendix 2]), for $p > 0$,

$$h(X) \le \log\left(\alpha_p \|X\|_p\right), \tag{2}$$

where

$$\alpha_p \triangleq 2e^{\frac{1}{p}}\Gamma\left(1+\frac{1}{p}\right)p^{\frac{1}{p}}, \tag{3}$$

$$\|X\|_p \triangleq \left(\mathbb{E}\left[|X|^p\right]\right)^{\frac{1}{p}}. \tag{4}$$

Of course, if $p = 2$, $\alpha_p = \sqrt{2\pi e}$, and (2) reduces to (1). A natural question to ask is whether a matching lower bound on $h(X)$ can be found in terms of $p$-norm of $X$, $\|X\|_p$. The quest is meaningless without additional assumptions on the density of $X$, as $h(X) = -\infty$ is possible even if $\|X\|_p$ is finite. In this paper, we show that if the density of $X$, $f_X(x)$, is *symmetric* (that is, $f_X(x) = f_X(-x)$) and *log-concave* (that is, $\log f_X(x)$ is concave), then $h(X)$ stays within a constant (dependent on $p$) from the upper bound in (2) (see Theorem 1 in Section II below):

$$h(X) \ge \log \frac{2\|X\|_p}{\Gamma(p+1)^{\frac{1}{p}}}, \tag{5}$$

where $\Gamma$ denotes the Gamma function, and $p > -1$, $p \neq 0$.

The class of log-concave random vectors is rich and contains important distributions in probability, statistics and analysis. Gaussian distribution, Laplace distribution, uniform distribution on a convex set, chi distribution are examples of log-concave distribution. Furthermore, a famous result of Prékopa [2] states that sums of log-concave random vectors, as well as marginals of random vectors, are log-concave. Thus, this class has good behavior under natural probabilistic operations.

Together with the classical bound in (2), the new bound in (5) tells us that entropy and moments of log-concave symmetric random variables are comparable.

A simple corollary to (5) is that if $X \ge 0$ has log-concave density with $\max_x f_X(x) = f_X(0)$, then

$$h(X) \ge \log \frac{\|X\|_p}{\Gamma(p+1)^{\frac{1}{p}}}. \tag{6}$$

The bound in (6) follows from (5) and the observation that for symmetric $X$, $h(X) = h(X|X \ge$


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$0) + \log 2$.

For $p = 2$, (5) reads

$$h(X) \geq \frac{1}{2} \log \left( 2\|X\|_2^2 \right). \tag{7}$$

Using a different proof technique, Bobkov and Madiman [3] recently showed that the differential entropy of a zero-mean non-symmetric log-concave $X$ satisfies,

$$h(X) \geq \frac{1}{2} \log \left( \frac{1}{2} \|X\|_2^2 \right). \tag{8}$$

Our result in (7) tightens (8) in the case of symmetric $X$, and our result in (6) tightens (8) in the case of $X$ with monotonic density.

The new bounds on the differential entropy are essential in the study of the difference between the rate-distortion function and the Shannon lower bound that we describe next. Given a nonnegative number $d$, the rate-distortion function $\mathbb{R}_X(d)$ under $r$-th moment distortion measure is defined as

$$\mathbb{R}_X(d) = \inf_{\substack{P_{\hat{X}|X}: \\ \mathbb{E}[|X - \hat{X}|^r] \leq d}} I(X; \hat{X}), \tag{9}$$

where the infimum is over all transition probability kernels $\mathbb{R} \mapsto \mathbb{R}$ satisfying the moment constraint. The celebrated Shannon lower bound [4] states that the rate-distortion function is lower bounded by

$$\mathbb{R}_X(d) \geq \underline{\mathbb{R}}_X(d) \triangleq h(X) - \log \left( \alpha_r d^{\frac{1}{r}} \right), \tag{10}$$

and $\alpha_r$ is defined in (3). For mean-square distortion ($r = 2$), (10) simplifies as

$$\mathbb{R}_X(d) \geq h(X) - \frac{1}{2} \log(2\pi e d). \tag{11}$$

The Shannon lower bound states that the rate-distortion function is lower bounded by the difference between the differential entropy of the source and the term that increases with target distortion $d$, explicitly linking the storage requirements for $X$ to the information content of $X$ (measured by $h(X)$) and the desired reproduction distortion $d$. As shown in [5]–[7] under progressively less stringent assumptions,[1] the Shannon lower bound is tight in the limit of low distortion,

$$0 \leq \mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d) \xrightarrow[d \to 0]{} 0. \tag{12}$$

[1]Koch [7] showed that (12) holds as long as $H(\lfloor X \rfloor) < \infty$.

The speed of convergence in (12) and its finite blocklength refinement were recently explored in [8], [9]. Due to its simplicity and tightness in the high resolution / low distortion limit, the Shannon lower bound can serve as a proxy for the rate-distortion function $\mathbb{R}_X(d)$, which rarely has an explicit representation. Furthermore, the tightness of the Shannon lower bound at low $d$ is linked to the optimality of simple lattice quantizers [8], [9], an insight which has evident practical significance. Gish and Pierce [10] showed that for mean-square error distortion, the difference between the entropy rate of a dithered scalar quantizer, $H_1$, and the rate-distortion function $\mathbb{R}_X(d)$ converges to $\frac{1}{2} \log \frac{2\pi e}{12} \approx 0.254$ bit/sample in the limit $d \downarrow 0$. Ziv [11] proved that $H_1 - \mathbb{R}_X(d)$ is bounded by $\frac{1}{2} \log \frac{2\pi e}{6} \approx 0.754$ bit/sample, universally in $d$.

In this paper, we show that the gap between $\mathbb{R}_X(d)$ and $\underline{\mathbb{R}}_X(d)$ is bounded universally in $d$, provided that the source density is symmetric and log-concave: for mean-square error distortion ($r = 2$ in (9)), we have

$$\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d) \leq \log \sqrt{\pi e} \approx 1.55 \text{ bits}. \tag{13}$$

Fig. 1 presents our bound for different values of $r$. Regardless of $r$ and $d$,

$$\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d) \leq \log(\sqrt{2} e) \approx 1.94 \text{ bits}. \tag{14}$$

The rest of the paper is organized as follows. Section II presents and discusses our main results: the lower bounds on differential entropy in Theorem 1, and the upper bound on $\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d)$ in Theorem 2. The convex geometry tools to prove the bounds on differential entropy are developed in Section III. The bound on the rate-distortion function is proven in Section IV.

The results presented here are part of a work in preparation [12]. In [12], we extend the results to non-symmetric log-concave random variables, and to higher dimension.

## II. Main results

A function $f : \mathbb{R}^n \to [0, +\infty)$ is *log-concave* if $\log(f) : \mathbb{R}^n \to [-\infty, \infty)$ is a concave function. Equivalently, $f$ is log-concave if for every $\lambda \in [0, 1]$ and for every $x, y \in \mathbb{R}^n$, one has

$$f((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}. \tag{15}$$

We say that a random vector $X$ in $\mathbb{R}^n$ is log-concave if it has a probability density function $f_X$

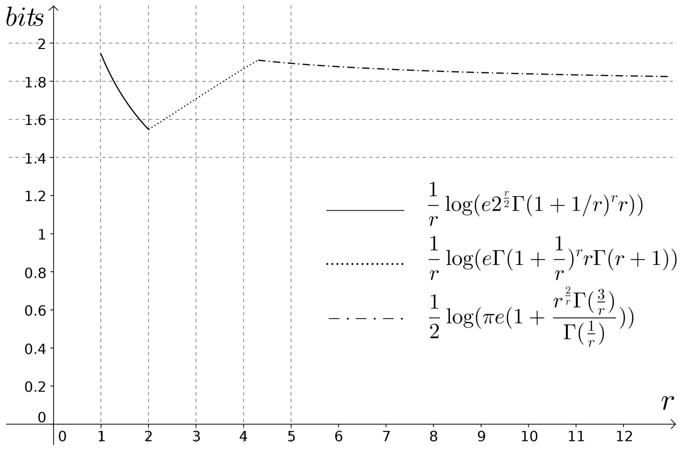


Fig. 1. The bound on the difference between the rate-distortion function under $r$-th moment constraint and the Shannon lower bound, stated in Theorem 2.

with respect to Lebesgue measure in $\mathbb{R}^n$ such that $f_X$ is log-concave. Our first result is a lower bound for the differential entropy of log-concave random variable in terms of the moments of the distribution.

**Theorem 1.** *Let $X$ be a symmetric log-concave random variable. Then, for every $p > -1$, $p \neq 0$,*

$$h(X) \geq \log \frac{2\|X\|_p}{\Gamma(p+1)^{\frac{1}{p}}}. \tag{16}$$

The difference between the upper bound in (2) and the lower bound in (16) grows as $\log(p)$ as $p \to +\infty$, as $\frac{1}{\sqrt{p}}$ as $p \to 0$, and is minimized at $p = 1$, the value being $\log(e) \approx 1.4$ bits.

The next proposition shows that the moments of a symmetric log-concave random variable are comparable, and demonstrates that the bound in Theorem 1 tightens as $p \downarrow -1$.

**Proposition 1.** *Let $X$ be a symmetric log-concave random variable. Then, for every $-1 < p \leq q$,*

$$\frac{\|X\|_q}{\Gamma(q+1)^{\frac{1}{q}}} \leq \frac{\|X\|_p}{\Gamma(p+1)^{\frac{1}{p}}}. \tag{17}$$

Combining Proposition 1 with the well known fact that $\|X\|_p$ is non-decreasing in $p$, we deduce that for every symmetric log-concave random variable $X$, for every $-1 < p < q$,

$$\frac{\|X\|_p}{\Gamma(q+1)^{\frac{1}{q}}} \leq \frac{\|X\|_q}{\Gamma(q+1)^{\frac{1}{q}}} \leq \frac{\|X\|_p}{\Gamma(p+1)^{\frac{1}{p}}}. \tag{18}$$

Using Theorem 1, we immediately obtain the following upper bound for the relative entropy $D(X||G_X)$ between a symmetric log-concave random variable $X$ and a Gaussian $G_X$ with same variance as that of $X$.

**Corollary 1.** *Let $X$ be a symmetric log-concave random variable. Then, for every $p > -1$, $p \neq 0$,*

$$D(X||G_X) \leq \log \sqrt{\pi e} + \Delta_p, \tag{19}$$

*where $G_X \sim \mathcal{N}(0, \|X\|_2^2)$, and*

$$\Delta_p \triangleq \log \left( \frac{\Gamma(p+1)^{\frac{1}{p}}}{\sqrt{2}} \frac{\|X\|_2}{\|X\|_p} \right). \tag{20}$$

*Remark* 1. The uniform distribution achieves equality in (19) in the limit $p \downarrow -1$. Indeed, if $X$ is uniformly distributed on a symmetric interval, then

$$\Delta_p = \log \frac{\Gamma(p+2)^{\frac{1}{p}}}{\sqrt{6}} \xrightarrow[p \to -1]{} \frac{1}{2} \log \frac{1}{6}. \tag{21}$$

Thus,

$$D(X||G_X) \leq \frac{1}{2} \log \frac{2\pi e}{12} = D(X||G_X). \tag{22}$$

Note that $\frac{1}{2\pi e}$ is the normalized second moment (i.e. the second moment per dimension of a uniform distribution divided by the volume raised to the power of $\frac{2}{n}$) of an $n$-dimensional ball, in the limit of large $n$, and $\frac{1}{12}$ is the normalized second moment of a hypercube.

*Remark* 2. For every symmetric log-concave random variable $X$, and for every $p \leq 2$, using Proposition 1, one has

$$\Delta_p \leq 0. \tag{23}$$

Thus, we necessarily have

$$D(X||G_X) \leq \frac{1}{2} \log(\pi e). \tag{24}$$

For a given distribution of $X$, one can optimize over $p$ to further tighten (24), as seen in (21) for the uniform distribution.

As an application of Theorem 1, we show in Theorem 2 below that in the class of 1-dimensional symmetric log-concave distributions, the rate distortion function does not exceed Shannon's lower bound by more than $\log(e\sqrt{2}) \approx 1.94$ bits, independently of $d$ and $r \geq 1$. Denote for brevity

$$\beta_r \triangleq \sqrt{1 + \frac{r^{\frac{2}{r}}\Gamma(\frac{3}{r})}{\Gamma(\frac{1}{r})}}. \tag{25}$$

and recall the definition of $\alpha_r$ in (3).

**Theorem 2.** *Let $d \geq 0$ and $r \geq 1$. Let $X$ be a symmetric log-concave random variable.*

1) *Let $r \in [1,2]$. If $\|X\|_2 > d^{\frac{1}{r}}$, then*

$$\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d) \leq D(X||G_X) + \log\frac{\alpha_r}{\sqrt{2\pi e}}. \tag{26}$$

*If $\|X\|_2 \leq d^{\frac{1}{r}}$, then $\mathbb{R}_X(d) = 0$.*

2) *Let $r > 2$. If $\|X\|_2 \geq d^{\frac{1}{r}}$, then*

$$\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d) \leq \tag{27}$$

$$D(X||G_X) + \min\left\{\log\frac{\alpha_r\Gamma(r+1)^{\frac{1}{r}}}{2\sqrt{\pi e}}, \ \log\beta_r\right\}.$$

*If $\|X\|_r \leq d^{\frac{1}{r}}$ or $\|X\|_2 \leq \frac{\sqrt{2}}{\Gamma(r+1)^{\frac{1}{r}}}d^{\frac{1}{r}}$, then $\mathbb{R}_X(d) = 0$.*

*If $\|X\|_r > d^{\frac{1}{r}}$ and $\|X\|_2 \in \left(\frac{\sqrt{2}}{\Gamma(r+1)^{\frac{1}{r}}}d^{\frac{1}{r}}, d^{\frac{1}{r}}\right)$, then*

$$\mathbb{R}_X(d) \leq \min\left\{\log\frac{\Gamma(r+1)^{\frac{1}{r}}}{\sqrt{2}}, \ \log\frac{\sqrt{2\pi e}\beta_r}{\alpha_r}\right\}.$$

For Gaussian $X$ and $r = 2$, the upper bound in (26) is 0, as expected. To bound $\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d)$ independently of the distribution of $X$, we apply the bound (24) on $D(X||G_X)$ to Theorem 2:

**Corollary 2.** *Let $X$ be a symmetric log-concave random variable. For $r \in [1,2]$, we have*

$$\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d) \leq \log\alpha_r - \frac{1}{2}\log 2. \tag{28}$$

*For $r > 2$, we have*

$$\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d)$$

$$\leq \ \min\left\{\log\frac{\alpha_r\Gamma(r+1)^{\frac{1}{r}}}{2}, \ \log\left(\sqrt{\pi e}\beta_r\right)\right\}. \tag{29}$$

Please refer to Fig. 1 in Section I for a numerical evaluation of the bounds of Corollary 2.

As mentioned in Remark 1, the bounds in Corollary 2 can be tightened by applying Corollary 1 with $p < 2$ to Theorem 2. For example, for mean-square distortion ($r = 2$) and a uniformly distributed source,

$$\mathbb{R}_X(d) - \underline{\mathbb{R}}_X(d) \leq \frac{1}{2}\log\frac{2\pi e}{12} \approx 0.254 \text{ bits}. \tag{30}$$

## III. Lower bounds on the Shannon entropy

In this section, we develop the main tools of the paper. The key to our development is the following result for 1-dimensional log-concave distributions, well known in convex geometry. It can be found in [13], in a slightly different form.

**Lemma 1** ([13])**.** *The function*

$$F(r) = \frac{1}{\Gamma(r+1)}\int_0^{+\infty} x^r f(x)\,dx \tag{31}$$

*is log-concave on $[-1,+\infty)$, whenever $f\colon [0;+\infty) \to [0;+\infty)$ is log-concave.*

Lemma 1 has been applied to obtain reverse entropy power inequalities [14], as well as to prove optimal concentration of the information content [15].

*Proof of Theorem 1.* Applying Lemma 1 to the values $-1, 0, p$, we have

$$F(0) = F\left(\frac{p}{p+1}(-1) + \frac{1}{p+1}p\right) \tag{32}$$

$$\geq F(-1)^{\frac{p}{p+1}}F(p)^{\frac{1}{p+1}}. \tag{33}$$

The bound in Theorem 1 will follow by computing the values $F(-1)$, $F(0)$ and $F(p)$ for $f = f_X$.

One has

$$F(0) = \frac{1}{2}, \tag{34}$$

$$F(p) = \frac{\|X\|_p^p}{2\Gamma(p+1)}. \tag{35}$$

To compute $F(-1)$, we first provide a different expression for $F(r)$. Notice that

$$F(r) = \frac{1}{\Gamma(r+1)}\int_0^{+\infty} x^r \int_0^{f_X(x)} dt\,dx \tag{36}$$

$$= \frac{r+1}{\Gamma(r+2)}\int_0^{\max f_X}\int_{\{x\geq 0\colon\, f_X(x)\geq t\}} x^r\,dx\,dt. \tag{37}$$

Denote the generalized inverse of $f_X$ by

$$f_X^{-1}(t) \triangleq \sup\{x \geq 0\colon\, f_X(x) \geq t\}, \quad t \geq 0. \tag{38}$$

Since $f_X$ is log-concave and

$$f_X(x) \leq f_X(0) = \max f_X, \tag{39}$$

it follows that $f_X$ is non-increasing on $[0,+\infty)$. Therefore,

$$\{x \geq 0\colon\, f_X(x) \geq t\} = [0, f_X^{-1}(t)]. \tag{40}$$

Hence,

$$F(r) = \frac{1}{\Gamma(r+2)} \int_0^{f_X(0)} (f_X^{-1}(t))^{r+1}\, dt. \quad (41)$$

We deduce that

$$F(-1) = f_X(0). \quad (42)$$

Plugging (34), (35) and (42) into (33), we obtain

$$f_X(0) \le \frac{\Gamma(p+1)^{\frac{1}{p}}}{2\|X\|_p}. \quad (43)$$

It follows immediately that

$$h(X) \ge \log \frac{1}{f_X(0)} \ge \log \frac{2\|X\|_p}{\Gamma(p+1)^{\frac{1}{p}}}. \quad (44)$$

For $p \in (-1, 0)$, the bound is obtained similarly by applying Lemma 1 to the values $-1, p, 0$. □

*Remark* 3. From (43) and (39), we see that a stronger statement than Theorem 1 holds: For every symmetric log-concave random variable $X \sim f_X$, for every $p > -1$, and for every $x \in \mathbb{R}$,

$$f_X(x) \le \frac{\Gamma(p+1)^{\frac{1}{p}}}{2\|X\|_p}. \quad (45)$$

*Proof of Proposition 1.* The proof is similar to the proof of Theorem 1. For example when $0 < p < q$, the bound is obtained by applying Lemma 1 to the values $0, p, q$. □

## IV. Bounds on the rate distortion function

We are now ready to prove Theorem 2. We will show part 1) of Theorem 2; the details of part 2) are carried out in [12].

*Proof of Theorem 2.* Denote for brevity $\sigma = \|X\|_2$. 1) Let $r \in [1,2]$. Assume $\sigma > d^{\frac{1}{r}}$. We take

$$\hat{X} = \left(1 - \frac{d^{\frac{2}{r}}}{\sigma^2}\right)(X+Z), \quad (46)$$

where $Z \sim \mathcal{N}\left(0, \frac{\sigma^2 d^{\frac{2}{r}}}{\sigma^2 - d^{\frac{2}{r}}}\right)$ is independent of $X$. This choice of $\hat{X}$ is admissible since

$$\begin{aligned} &\mathbb{E}\left[|X-\hat{X}|^r\right] \\ &\le \mathbb{E}\left[|X-\hat{X}|^2\right]^{\frac{r}{2}} && (47) \\ &\le \left[\left(\frac{d^{\frac{2}{r}}}{\sigma^2}\right)^2 \sigma^2 + \left(1-\frac{d^{\frac{2}{r}}}{\sigma^2}\right)^2 \mathbb{E}\left[|Z|^2\right]\right]^{\frac{r}{2}} && (48) \\ &= d, && (49) \end{aligned}$$

where we used $r \le 2$ and the left-hand side of inequality (18). Hence,

$$\begin{aligned} \mathbb{R}_X(d) \le I(X;\hat{X}) &= h(\hat{X}) - h(\hat{X}|X) && (50) \\ &= h(X+Z) - h(Z), && (51) \end{aligned}$$

where we used homogeneity of entropy for the last equality. Invoking the upper bound on the differential entropy (1), we have

$$\begin{aligned} &h(X+Z) - h(Z) \\ &\le \frac{1}{2}\log\left(2\pi e\left(\sigma^2 + \frac{\sigma^2 d^{\frac{2}{r}}}{\sigma^2 - d^{\frac{2}{r}}}\right)\right) - h(Z) && (52) \\ &= \frac{1}{2}\log\frac{\sigma^2}{d^{\frac{2}{r}}} && (53) \\ &= \mathbb{R}_X(d) + D(X||G_X) + \log\frac{\alpha_r}{\sqrt{2\pi e}}, && (54) \end{aligned}$$

and (26) follows.

If $\|X\|_2 \le d^{\frac{1}{r}}$, then $\|X\|_r \le \|X\|_2 \le d^{\frac{1}{r}}$, and setting $\hat{X} \equiv 0$ leads to $\mathbb{R}_X(d) = 0$. □

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