# A Rényi entropy power inequality for log-concave vectors and parameters in $[0, 1]$


Arnaud Marsiglietti*
Center for the Mathematics of Information
California Institute of Technology
Pasadena, CA 91125, USA
amarsigl@caltech.edu

James Melbourne**
Electrical and Computer Engineering
University of Minnesota
Minneapolis, MN 55455, USA
melbo013@umn.edu



***Abstract*—Using a sharp version of the reverse Young inequality, and a Rényi entropy comparison result due to Fradelizi, Madiman, and Wang, the authors derive a Rényi entropy power inequality for log-concave random vectors when Rényi parameters belong to $[0, 1]$. A discussion of symmetric decreasing rearrangements of random variables strengthens the inequality and guides the exploration as to its sharpness.**



**Keywords.** Entropy power inequality, Rényi entropy, Log-concave.


## I. Introduction

The Rényi entropy [31] of parameter $r \in [0, \infty]$ is defined for continuous random vectors $X$ with density $f_X$ as

$$h_r(X) = \frac{1}{1-r} \log \left( \int_{\mathbb{R}^n} f_X^r(x)\, dx \right).$$

We take the Rényi entropy power of $X$ to be

$$N_r(X) = e^{\frac{2}{n} h_r(X)} = \left( \int_{\mathbb{R}^n} f_X(x)^r\, dx \right)^{-\frac{2}{n}\frac{1}{r-1}}.$$

Three important cases are handled by continuous limits,

$$N_0(X) = \mathrm{Vol}^{\frac{2}{n}}(\mathrm{supp}(X)),$$

$$N_\infty(X) = \|f_X\|_\infty^{-2/n},$$

and $N_1(X)$ corresponds to the usual Shannon entropy power $N_1(X) = N(X) = e^{-\frac{2}{n}\int f_X \log f_X}$. Here, $\mathrm{Vol}(A)$ denotes the volume of $A$, and $\mathrm{supp}(X)$ denotes the support of $X$.

The entropy power inequality [32], [33] (EPI) is the statement that Shannon entropy power of independent random vectors $X$ and $Y$ is super-additive

$$N(X + Y) \geq N(X) + N(Y).$$

In this language we interpret the Brunn-Minkowski inequality of Convex Geometry, classically stated as the fact that

$$\mathrm{Vol}^{\frac{1}{n}}(A + B) \geq \mathrm{Vol}^{\frac{1}{n}}(A) + \mathrm{Vol}^{\frac{1}{n}}(B)$$

for any pair of compact sets of $\mathbb{R}^n$, as a Rényi-EPI corresponding to $r = 0$. That is, the Brunn-Minkowski inequality is equivalent to the fact that for $X$ and $Y$ independent random vectors, the square root of the 0-th Rényi entropy is super-additive,

$$N_0^{\frac{1}{2}}(X + Y) \geq N_0^{\frac{1}{2}}(X) + N_0^{\frac{1}{2}}(Y).$$

The sharp version of Young's inequality was used by Brascamp and Lieb [12] to derive a proof of the Brunn-Minkowski inequality, and soon after Lieb alone used the same machinery again to give a proof of the entropy power inequality [24]. In [13] Costa and Cover brought attention to the analogies between the two inequalities and Dembo, Cover and Thomas [15] observed that the two Young's inequality proofs of [12], [24] could be unified.

The authors give a more thorough narration of the developments that inspired this work in [29], interesting results in this direction can be found in [1], [2], [7], [8], [13], [15], [17], [18], [20], [23], [25], [36], [38]. Previous work on Rényi EPIs in the case that $r \in \{0\} \cup [1, \infty]$ can be found in [5], [6], [9], [22], [27], [30], [39]–[42].

The existence of super-additivity properties of the Rényi entropy power for $r \in (0, 1)$ had been mentioned as an open problem in [6], [22], [26], [30]. In this article, we summarize the contributions of a recent work by the authors in [29], where a Rényi EPI is derived in the log-concave case (see Definition 3) for a modified exponent. This main result is the following.

**Theorem 1.** *Let $r \in (0, 1)$. Let $X, Y$ be log-concave random vectors in $\mathbb{R}^n$. Then,*

$$N_r(X + Y)^\alpha \geq N_r(X)^\alpha + N_r(Y)^\alpha, \qquad (1)$$


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*where*

$$\alpha = \alpha(r) \triangleq \frac{(1-r)\log 2}{(1+r)\log(1+r) + r\log\frac{1}{4r}}.$$

It would be of interest to extend Theorem 1 to general independent $X$ and $Y$, and to determine the sharp exponent $\alpha_{opt} = \alpha_{opt}(r)$, the infimum over all $\alpha$ that satisfy

$$N_r(X+Y)^\alpha \geq N_r(X)^\alpha + N_r(Y)^\alpha, \tag{2}$$

for any pair of independent random vectors $X$ and $Y$.

However even under the restriction to log-concave random variables, Theorem 1 cannot be significantly improved. Indeed we will show by computations on a pair of independent Laplace distributed random variables that we must have

$$\alpha(r) \geq \alpha_{opt}(r) \geq \max\left(1, \frac{(1-r)\log 2}{2(\log E(r) - r\log 2)}\right). \tag{3}$$

where we have used the notation

$$E(r) \triangleq \int_0^\infty \left(1+\frac{x}{r}\right)^r e^{-x}dx. \tag{4}$$

As we will argue this necessarily tightens the bound

$$\frac{(1-r)\log 2}{2\log\Gamma(1+r) + 2r\log\frac{1}{r}} \leq \alpha_{opt} \tag{5}$$

derived in [29]. Note that the above bounds imply that

$$\lim_{r\to 1}\alpha(r) = \lim_{r\to 1}\alpha_{opt}(r) = 1,$$

recovering the usual EPI, however as $r \to 0$ the behavior of general log-concave vectors is much worse than convex bodies ($r = 0$), and

$$\lim_{r\to 0}\alpha(r)r^{1-\varepsilon} = \lim_{r\to 0}\alpha_{opt}(r)r^{1-\varepsilon} = +\infty,$$

for any $\varepsilon > 0$. This gives a striking discontinuity of $\alpha_{opt}$ at $r = 0$, since $\alpha_{opt}(0) = \frac{1}{2}$ by the Brunn-Minkowski inequality. Although these bounds preclude the possibility of a smooth interpolation of Rényi entropy power inequalities between the Classical EPI and the Brunn-Minkowski inequality for general random variables, in the case that random variables are uniform distributions on compact sets (not necessarily convex) we have the following.

**Theorem 2.** *Let $r \in (0,1)$. Let $X, Y$ be uniformly distributed random vectors on compact sets. Then,*

$$N_r(X+Y)^\beta \geq N_r(X)^\beta + N_r(Y)^\beta,$$

*where*

$$\beta = \beta(r) \triangleq \frac{(1-r)\log 2}{2\log 2 + r\log r - (r+1)\log(r+1)}.$$

See [29] for proof. Notice that $\lim_{r\to 0}\beta(r) = \frac{1}{2}$ recovering the Brunn-Minkowski inequality, while predictably $\lim_{r\to 1}\beta(r) = 1$ gives a special case of the entropy power inequality for uniform distributions.

In Section II, we present the main tools in establishing Theorems 1 and 2. The proof of Theorem 1 is given in Section III. In Section IV, we derive a lower bound on the optimal exponent $\alpha_{opt}$ that satisfies (2). We compare our results with properties of the symmetric decreasing rearrangements and improve on previous lower bounds for $\alpha_{opt}$. The details of all omitted proofs can be found in [29].

## II. Preliminaries

For $p \in [0,\infty]$, we denote by $p'$ the conjugate of $p$,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

For a non-negative function $f : \mathbb{R}^n \to [0,+\infty)$ we introduce the notation

$$\|f\|_p = \left(\int_{\mathbb{R}^n} f^p(x)dx\right)^{1/p}.$$

**Definition 3.** *A random vector $X$ in $\mathbb{R}^n$ is log-concave if it possesses a log-concave density $f_X : \mathbb{R}^n \to [0,+\infty)$ with respect to Lebesgue measure. That is that for all $\lambda \in (0,1)$ and $x, y \in \mathbb{R}^n$,*

$$f_X((1-\lambda)x + \lambda y) \geq f_X^{1-\lambda}(x) f_X^\lambda(y).$$

Log-concave random vectors and functions are important classes in many disciplines. In the context of information theory, several nice properties involving entropy of log-concave random vectors were recently established (see, e.g., [7], [3], [34], [35], [14], [28]). Significant examples are Gaussian and exponential distributions as well as any uniform distribution on a convex set. We direct the interested reader to [26] for more background on their role in information theory and convex geometry.

The main tool in establishing Theorems 1 and 2 is the reverse form of the sharp Young inequality. The reversal of Young's inequality for parameters in $[0,1]$ is due to Leindler [21], while sharp constants were obtained independently by Beckner [4], and Brascamp and Lieb [12]:

**Theorem 4** ([4], [12])**.** *Let $0 \leq p, q, r \leq 1$ such that $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$. Then,*

$$\|f \star g\|_r \geq C^{\frac{n}{2}} \|f\|_p \|g\|_q, \tag{6}$$

*where*

$$C = C(p,q,r) = \frac{c_p c_q}{c_r}, \qquad c_m = \frac{m^{1/m}}{|m'|^{1/m'}}.$$

Given independent random vectors $X$ with density $f$ and $Y$ with density $g$, the random vector $X+Y$ will be distributed according to $f \star g$. Observe that the $L_p$ "norms" have the following expression as Rényi entropy powers, $\|f\|_r = N_r(X)^{-\frac{n}{2r'}}$. Hence, we can rewrite (6) as follows,

$$N_r(X+Y)^{-\frac{1}{r'}} \geq C N_p(X)^{-\frac{1}{p'}} N_q(Y)^{-\frac{1}{q'}}. \tag{7}$$

This is an information-theoretic interpretation of Young's inequality, which was developed in [15].

We also need a Rényi comparison result for log-concave random vectors, implicit in [19], and with a generalized version to appear in [16] for the $s$-concave measures (see [10], [11] for more background on this generalization of log-concavity).

**Lemma 5** ([16])**.** *Let $0 < p < q$. Then, for every log-concave random vector $X$,*

$$N_q(X) \leq N_p(X) \leq \frac{p^{\frac{2}{p-1}}}{q^{\frac{2}{q-1}}} N_q(X).$$

The first inequality is classical and holds for general $X$, a consequence of the fact that $N_p(X)$ can be expressed as the reciprocal of a $p-1$ norm $\left(\mathbb{E} f^{p-1}(X)\right)^{-1/(p-1)}$. The increasingness of norms (which follows from Jensen's inequality) implies the decreasingness of Rényi entropy powers. The content of Fradelizi, Madiman, and Wang's result is thus the second inequality, that this decrease is not too rapid for log-concave random vectors. A proof of Lemma 5 can be found in the appendix of [29].

## III. Proof of Theorem 1

We first combine the information-theoretic formulation of reverse Young's inequality (7) and Lemma 5 to obtain,

$$\begin{aligned} &N_r(X+Y)^{-\frac{1}{r'}} \\ &\geq C \left( \frac{p^{\frac{2}{p-1}}}{r^{\frac{2}{r-1}}} \right)^{-\frac{1}{p'}} \left( \frac{q^{\frac{2}{q-1}}}{r^{\frac{2}{r-1}}} \right)^{-\frac{1}{q'}} N_r(X)^{-\frac{1}{p'}} N_r(Y)^{-\frac{1}{q'}} \\ &= A(p,q,r) N_r(X)^{-\frac{1}{p'}} N_r(Y)^{-\frac{1}{q'}}, \end{aligned} \tag{8}$$

where

$$A(p,q,r) = \frac{c_p c_q}{c_r} \frac{r^{\frac{2}{r}}}{p^{\frac{2}{p}} q^{\frac{2}{q}}}.$$

Equivalently,

$$N_r(X+Y) \geq A(p,q,r)^{-r'} N_r(X)^{\frac{r'}{p'}} N_r(Y)^{\frac{r'}{q'}}. \tag{9}$$

Thus to complete our proof of Theorem 1 it suffices to obtain for a fixed $r \in (0,1)$, an $\alpha > 0$ such that for any given pair of independent log-concave random vectors $X$ and $Y$, there exist $0 \leq p, q \leq 1$ such that $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$ and

$$\begin{aligned} A(p,q,r)^{-\alpha r'} N_r(X)^{\frac{\alpha r'}{p'}} N_r(Y)^{\frac{\alpha r'}{q'}} \\ \geq N_r^{\alpha}(X) + N_r^{\alpha}(Y). \end{aligned} \tag{10}$$

Let us observe that there is nothing probabilistic about equation (10). If we write $x = N_r(X)^{\alpha}$, $y = N_r(Y)^{\alpha}$, our Rényi entropy power inequality is implied by the following algebraic inequality.

**Proposition 6.** *Given $r \in (0,1)$ and taking*

$$\alpha = \frac{(1-r)\log 2}{(1+r)\log(1+r) + r \log \frac{1}{4r}}, \tag{11}$$

*then for any $x, y > 0$ there exist $0 < p, q < 1$ satisfying $\frac{1}{p'} + \frac{1}{q'} = \frac{1}{r'}$ such that*

$$A(p,q,r)^{-\alpha r'} x^{\frac{r'}{p'}} y^{\frac{r'}{q'}} \geq x + y. \tag{12}$$

The proof of Proposition 6 can be found in [29].

## IV. Lower bound on the optimal exponent

Recalling $\alpha_{opt} = \alpha_{opt}(r)$ the infimum over all $\alpha$ satisfying the Rényi entropy power inequality (2). Theorem 1 gives upper bounds on the optimal $\alpha$ satisfying the super-additivity condition when we restrict to the class of log-concave random vectors. Conversely, one can derive lower bounds on $\alpha_{opt}$ by testing well chosen examples.

By simply choosing $Z_1, Z_2$ i.i.d. standard Gaussians, we have by homogeneity of Rényi entropy,

$$N_r^{\alpha_{opt}}(Z_1 + Z_2) = 2^{\alpha_{opt}} N_r^{\alpha_{opt}}(Z_1),$$

while

$$N_r^{\alpha_{opt}}(Z_1) + N_r^{\alpha_{opt}}(Z_2) = 2 N_r^{\alpha_{opt}}(Z_1).$$

It follows that

$$\alpha_{opt} \geq 1. \tag{13}$$

Though this is already strictly greater than the $\alpha_{opt}(0) = 1/2$ achievable by the Brunn-Minkowski inequality, for other log-concave distributions the behavior can be much worse. Indeed by direct computation (see [29]) on $X$ and $Y$ i.i.d. exponential on $(0,\infty)$ it follows that

$$\alpha_{opt} \geq \frac{(1-r)\log 2}{2\log\Gamma(r+1)+2r\log\frac{1}{r}}. \tag{14}$$

Let us mention that although all the computations here are done on one-dimensional examples $X$ and $Y$, they can be easily extended to $n$-dimensions by taking $\tilde{X}=(X_1,\ldots,X_n)$ and $\tilde{Y}=(Y_1,\ldots,Y_n)$ with $X_i$ and $Y_j$ independent copies of $X$ and $Y$ respectively, then $N_r(X+Y)=N_r(\tilde{X}+\tilde{Y})$, $N_r(X)=N_r(\tilde{X})$, and $N_r(Y)=N_r(\tilde{Y})$.

In what follows we establish a tighter lower bound for $\alpha_{opt}$ than (14). Due to Madiman and Wang [37] it is known that the symmetric decreasing rearrangement of independent random vectors never increases the Rényi entropy of their sum, while preserving the Rényi entropy of the individual random variables. To state this more explicitly, we need a few definitions. For a measurable set $A$, denote by $A^*$ the open origin symmetric Euclidean ball satisfying $\mathrm{Vol}(A)=\mathrm{Vol}(A^*)$. For a non-negative measurable function $f$, define its symmetric decreasing rearrangement by

$$f^*(x)=\int_0^\infty \mathbb{1}_{\{f>t\}^*}(x)dt.$$

Finally, we can define $X^*$.

**Definition 7.** *For a random vector $X$ with density $f$, we denote by $X^*$ a random vector with density $f^*$.*

**Theorem 8** ( [37])**.** *For $X$ and $Y$ independent, with $X^*$ and $Y^*$ drawn independently from the symmetric decreasing rearrangements,*

$$N_r(X+Y)\geq N_r(X^*+Y^*).$$

Thus to prove a Rényi EPI for all independent random vectors it suffices to prove the result for random vectors with symmetrically decreasing densities. Combining this with our results, it follows that the conclusion of Theorem 1 actually holds for a more general class of random vectors, namely those with log-concave symmetrically decreasing rearrangement.

**Theorem 9.** *Suppose $X$ and $Y$ are independent random vectors such that $X^*$ and $Y^*$ are log-concave, then*

$$N_r^\alpha(X+Y)\geq N_r^\alpha(X)+N_r^\alpha(Y).$$

*Proof.* Drawing $X^*$ and $Y^*$ to be independent, we have

$$\begin{aligned} N_r^\alpha(X+Y) &\geq N_r^\alpha(X^*+Y^*)\\ &\geq N_r^\alpha(X^*)+N_r^\alpha(Y^*)\\ &= N_r^\alpha(X)+N_r^\alpha(Y). \end{aligned}$$

The first inequality is by Theorem 8, and the second by Theorem 1 applied to $X^*$ and $Y^*$. The last equality is due to the equimeasurability of densities and their rearrangements. □

Again with the rearrangement results of Madiman and Wang in mind, it is more appropriate to replace the exponential distributions leading to (14) with their symmetric decreasing rearrangements, Laplace distributions. Indeed, we will see that this does in fact improve bounds on $\alpha_{opt}$. Take $Y_i$, $i=1,2$, with density

$$e^{-|x|}/2 \quad (x\in\mathbb{R}).$$

In this case it is still straightforward to compute

$$N_r(Y_i)=\left(\frac{2^{1-r}}{r}\right)^{\frac{2}{1-r}},$$

while $Y_1+Y_2$ has density

$$(1+|x|)e^{-|x|}/4 \quad (x\in\mathbb{R})$$

when $Y_i$ are independent. Direct computation gives,

$$N_r(Y_1+Y_2)=\left(2^{1-2r}E(r)/r\right)^{\frac{2}{1-r}}.$$

where $E(r)$ is defined as in (4). Since

$$N_r^{\alpha_{opt}}(Y_1+Y_2)\geq N_r^{\alpha_{opt}}(Y_1)+N_r^{\alpha_{opt}}(Y_2)$$

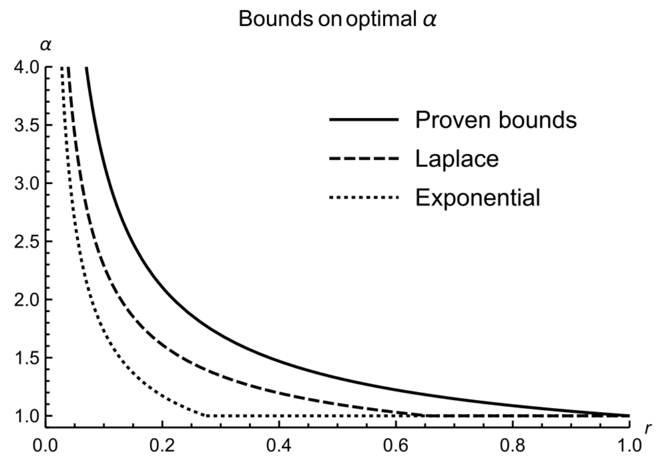
holds, it must be the case that

$$\left(2^{1-2r}E(r)/r\right)^{2\alpha_{opt}/(1-r)}\geq 2\left(\frac{2^{1-r}}{r}\right)^{2\alpha_{opt}/(1-r)}.$$

Taking logarithms, this rearranges to

$$\alpha_{opt}\geq\frac{(1-r)\log 2}{2(\log E(r)-r\log 2)}.$$

We can summarize these bounds in the following graphic, where the relationship $\alpha(r)$ in the Rényi EPI derived here for log-concave vectors, is compared with the lower bounds on $\alpha_{opt}$ induced by the exponential and Laplace distributions. These curves are cut off at 1 where they no longer give improvement on the Gaussian bounds derived earlier.




## References


[1] S. Artstein, K. M. Ball, F. Barthe, and A. Naor. Solution of Shannon's problem on the monotonicity of entropy. *J. Amer. Math. Soc.*, 17(4):975–982 (electronic), 2004.

[2] K. Ball, P. Nayar, and T. Tkocz. A reverse entropy power inequality for log-concave random vectors. *Studia Mathematica*, 235:17–30, 2016.

[3] K. Ball and V. H. Nguyen. Entropy jumps for isotropic log-concave random vectors and spectral gap. *Studia Math.*, 213(1):81–96, 2012.

[4] W. Beckner. Inequalities in Fourier analysis. *Ann. of Math. (2)*, 102(1):159–182, 1975.

[5] S. G. Bobkov and G. P. Chistyakov. Bounds for the maximum of the density of the sum of independent random variables. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 408(Veroyatnost i Statistika. 18):62–73, 324, 2012.

[6] S. G. Bobkov and G. P. Chistyakov. Entropy power inequality for the Rényi entropy. *IEEE Trans. Inform. Theory*, 61(2):708–714, February 2015.

[7] S. G. Bobkov and M. Madiman. The entropy per coordinate of a random vector is highly constrained under convexity conditions. *IEEE Trans. Inform. Theory*, 57(8):4940–4954, August 2011.

[8] S. G. Bobkov and M. Madiman. Reverse Brunn-Minkowski and reverse entropy power inequalities for convex measures. *J. Funct. Anal.*, 262:3309–3339, 2012.

[9] S. G. Bobkov and A. Marsiglietti. Variants of the entropy power inequality. *IEEE Transactions on Information Theory*, 63(12):7747–7752, 2017.

[10] S. G. Bobkov and J. Melbourne. Hyperbolic measures on infinite dimensional spaces. *Probability Surveys*, 13:57–88, 2016.

[11] C. Borell. Convex measures on locally convex spaces. *Ark. Mat.*, 12:239–252, 1974.

[12] H. J. Brascamp and E. H. Lieb. Best constants in Young's inequality, its converse, and its generalization to more than three functions. *Advances in Math.*, 20(2):151–173, 1976.

[13] M. H. M. Costa and T. M. Cover. On the similarity of the entropy power inequality and the Brunn-Minkowski inequality. *IEEE Trans. Inform. Theory*, 30(6):837–839, 1984.

[14] T. A. Courtade, M. Fathi, and A. Pananjady. Wasserstein stability of the entropy power inequality for log-concave densities. *Preprint, arXiv:1610.07969*, 2016.

[15] A. Dembo, T. M. Cover, and J. A. Thomas. Information-theoretic inequalities. *IEEE Trans. Inform. Theory*, 37(6):1501–1518, 1991.

[16] M. Fradelizi, J. Li, and M. Madiman. Concentration of information content for convex measures. *Preprint,* arXiv:1512.01490, 2015.

[17] M. Fradelizi, M. Madiman, A. Marsiglietti, and A. Zvavitch. The convexification effect of Minkowski summation. *Preprint, arXiv:1704.05486*, 2016.

[18] M. Fradelizi, M. Madiman, A. Marsiglietti, and A. Zvavitch. Do Minkowski averages get progressively more convex? *C. R. Acad. Sci. Paris Sér. I Math.*, 354(2):185–189, February 2016.

[19] M. Fradelizi, M. Madiman, and L. Wang. Optimal concentration of information content for log-concave densities. In *High dimensional probability VII*, pages 45–60. Springer, 2016.

[20] M. Fradelizi and A. Marsiglietti. On the analogue of the concavity of entropy power in the Brunn-Minkowski theory. *Adv. in Appl. Math.*, 57:1–20, 2014.

[21] L. Leindler. On a certain converse of Hölder's inequality. II. *Acta Sci. Math. (Szeged)*, 33(3-4):217–223, 1972.

[22] J. Li. Rényi entropy power inequality and a reverse. *Studia Mathematica*, 242:303–319, February 2018.

[23] J. Li and J. Melbourne. Further investigations of the maximum entropy of the sum of two dependent random variables. In *Proc. IEEE Intl. Symp. Inform. Theory.*, Vail, Colorado, 2018.

[24] E. H. Lieb. Proof of an entropy conjecture of Wehrl. *Comm. Math. Phys.*, 62(1):35–41, 1978.

[25] M. Madiman and A. R. Barron. Generalized entropy power inequalities and monotonicity properties of information. *IEEE Trans. Inform. Theory*, 53(7):2317–2329, July 2007.

[26] M. Madiman, J. Melbourne, and P. Xu. Forward and reverse entropy power inequalities in convex geometry. *Convexity and Concentration*, pages 427–485, 2017.

[27] M. Madiman, J. Melbourne, and P. Xu. Rogozin's convolution inequality for locally compact groups. *Preprint, arXiv:1705.00642*, 2017.

[28] A. Marsiglietti and V. Kostina. A lower bound on the differential entropy of log-concave random vectors with applications. *Entropy*, 20(3):185, 2018.

[29] A. Marsiglietti and J. Melbourne. On the entropy power inequality for the Rényi entropy of order [0,1]. *Preprint, arXiv:1710.00800*, 2017.

[30] E. Ram and I. Sason. On Rényi entropy power inequalities. *IEEE Transactions on Information Theory*, 62(12):6800–6815, 2016.

[31] A. Rényi. On measures of entropy and information. In *Proc. 4th Berkeley Sympos. Math. Statist. and Prob., Vol. I*, pages 547–561. Univ. California Press, Berkeley, Calif., 1961.

[32] C. E. Shannon. A mathematical theory of communication. *Bell System Tech. J.*, 27:379–423, 623–656, 1948.

[33] A. J. Stam. Some inequalities satisfied by the quantities of information of Fisher and Shannon. *Information and Control*, 2:101–112, 1959.

[34] G. Toscani. A strengthened entropy power inequality for log-concave densities. *IEEE Trans. Inform. Theory*, 61(12):6550–6559, 2015.

[35] G. Toscani. A concavity property for the reciprocal of Fisher information and its consequences on Costa's EPI. *Phys. A*, 432:35–42, 2015.

[36] A. M. Tulino and S. Verdú. Monotonic decrease of the non-gaussianness of the sum of independent random variables: A simple proof. *IEEE Trans. Inform. Theory*, 52(9):4295–7, September 2006.

[37] L. Wang and M. Madiman. Beyond the entropy power inequality, via rearrangements. *IEEE Trans. Inform. Theory*, 60(9):5116–5137, September 2014.

[38] P. Xu, J. Melbourne, and M. Madiman. Reverse entropy power inequalities for $s$-concave densities. In *Proc. IEEE Intl. Symp. Inform. Theory.*, pages 2284–2288, Barcelona, Spain, July 2016.

[39] P. Xu, J. Melbourne, and M. Madiman. A min-entropy power inequality for groups . In *Proc. IEEE Intl. Symp. Inform. Theory.*, 2017.

[40] P. Xu, J. Melbourne, and M. Madiman. Infinity entropy power inequalities. In *Proc. IEEE Intl. Symp. Inform. Theory.*, 2017.

[41] R. Zamir and M. Feder. A generalization of the entropy power inequality with applications. *IEEE Trans. Inform. Theory*, 39(5):1723–1728, 1993.

[42] R. Zamir and M. Feder. On the volume of the Minkowski sum of line sets and the entropy-power inequality. *IEEE Trans. Inform. Theory*, 44(7):3039–3063, 1998.