

THÈSE

Pour l'obtention du grade de

DOCTEUR ÈS SCIENCES

ÉCOLE DOCTORALE MATHÉMATIQUES ET SCIENCES ET TECHNOLOGIE
DE L'INFORMATION ET DE LA COMMUNICATION

DISCIPLINE : MATHÉMATIQUES

Présentée par

Arnaud MARSIGLIETTI

Intitulée

**Géométrie des mesures convexes et liens avec la
théorie de l'information**

Dirigée par **Mathieu FRADELIZI**

Soutenue le **mardi 24 juin 2014**

Devant le **jury** composé de

Jean-François BERCHER	Univ. Paris-Est Marne-la-Vallée	Examineur
Sergey BOBKOV	University of Minnesota	Rapporteur
François BOLLEY	Univ. Paris-Dauphine	Examineur
Dario CORDERO-ERAUSQUIN	Univ. Pierre et Marie Curie	Examineur
Mathieu FRADELIZI	Univ. Paris-Est Marne-la-Vallée	Directeur de thèse
Ivan GENTIL	Univ. Claude Bernard Lyon 1	Rapporteur
Olivier GUÉDON	Univ. Paris-Est Marne-la-Vallée	Examineur
Mathieu MEYER	Univ. Paris-Est Marne-la-Vallée	Président du jury

Thèse préparée au

Laboratoire d'Analyse et de Mathématiques Appliquées

LAMA UMR 8050 CNRS

Université Paris-Est Marne-la-Vallée

5, boulevard Descartes, Champs-sur-Marne

77454 Marne-la-Vallée cedex 2

FRANCE

Résumé

Ma thèse est consacrée à l'étude des mesures convexes ainsi qu'à l'étude des liens entre la théorie de Brunn-Minkowski et la théorie de l'information de Shannon.

Je poursuis les travaux de Costa et Cover qui ont mis en lumière des similitudes entre deux grandes théories mathématiques, la théorie de Brunn-Minkowski d'une part et la théorie de l'information d'autre part. Partant de ces similitudes, ils ont conjecturé, comme analogue de la concavité de l'entropie exponentielle, que la racine n -ième du volume parallèle de tout ensemble compact de \mathbb{R}^n est une fonction concave, et je résous cette conjecture de manière détaillée. Par ailleurs, j'étudie les mesures convexes définies par Borell et je démontre, pour ces mesures, une inégalité renforcée de type Brunn-Minkowski pour les ensembles convexes symétriques.

Cette thèse se décompose en quatre parties. Tout d'abord, je rappelle un certain nombre de notions de base concernant la théorie de Brunn-Minkowski et la théorie de l'information. Dans une seconde partie, j'établis la validité de la conjecture de Costa-Cover sous certaines conditions et je démontre qu'en toute généralité, cette conjecture est fautive en exhibant des contre-exemples explicites. Dans une troisième partie, j'étends les résultats positifs de cette conjecture de deux manières, d'une part en généralisant la notion de volume et d'autre part en établissant des versions fonctionnelles. Enfin, je prolonge des travaux récents de Gardner et Zvavitch en améliorant la concavité des mesures convexes sous certaines hypothèses telles que la symétrie.

Mots clés : (B)-conjecture, Brunn-Minkowski, Entropie, Entropie exponentielle, Entropie de Rényi, Equation de Hamilton-Jacobi, Géométrie convexe, Isopérimétrie, Localisation, Mesure convexe, Mesure gaussienne, Théorie de l'information, Volume parallèle.

Title

Geometry of Convex Measures and Links with the Information Theory

Abstract

This thesis is devoted to the study of convex measures as well as the relationships between the Brunn-Minkowski theory and the Information theory.

I continue the works of Costa and Cover who highlighted similarities between two fundamental inequalities in the Brunn-Minkowski theory and in the Information theory. Starting with these similarities, they conjectured, as an analogue of the concavity of entropy power, that the n -th root of the parallel volume of every compact subset of \mathbb{R}^n is concave, and I give a complete answer to this conjecture. I also study the convex measures defined by Borell, and for these measures I establish a refined inequality of the Brunn-Minkowski type when restricted to convex symmetric sets.

This thesis is split into four parts. In the first part, I recall some basic facts about the Brunn-Minkowski theory and the Information theory. In the second part, I prove the validity of the conjecture of Costa-Cover under special conditions and I show that the conjecture is false in such a generality by giving explicit counterexamples. In the third part, I extend the positive results of this conjecture by considering measures other than the classical volume and by establishing functional versions. Finally, in the fourth part I generalize recent works of Gardner and Zvavitch by improving the concavity of convex measures under different kinds of hypothesis such as symmetries.

Keywords: (B)-conjecture, Brunn-Minkowski, Convex geometry, Convex measure, Entropy, Entropy power, Rényi entropy, Gaussian measure, Hamilton-Jacobi equation, Isoperimetry, Information theory, Localization, Parallel volume.

Travaux

Je donne ici une liste de mes publications effectuées durant ma thèse et qui sont disponibles sur ma page internet personnelle, ainsi que sur les archives ouvertes HAL et arXiv.

Liste de publications

1. *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory*, (avec Matthieu Fradelizi), *Advances in Applied Mathematics* 57 (2014), 1-20.
2. *Concavity properties of extensions of the Parallel volume*, *Mathematika* 62 (2016), no. 1, 266-282.
3. *On the improvement of concavity of convex measures*, *Proceedings of the American Mathematical Society* 144 (2016), no. 2, 775-786.

List of Publications

Here is the list of research articles I completed during my Ph.D., which are available on my personal web page, or on the open archives HAL and arXiv.

Publication list

1. *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory*, (with Matthieu Fradelizi), *Advances in Applied Mathematics* 57 (2014), 1-20.
2. *Concavity properties of extensions of the Parallel volume*, *Mathematika* 62 (2016), no. 1, 266-282.
3. *On the improvement of concavity of convex measures*, *Proceedings of the American Mathematical Society* 144 (2016), no. 2, 775-786.

Remerciements

Il est maintenant temps pour moi de rédiger ce qui a été trois années de découvertes et d'immersion dans un monde sans limites qu'est la recherche mathématique. Avant d'entrer dans le détail de mes travaux effectués durant ma thèse, je souhaite écrire quelques mots de remerciements aux personnes qui m'ont accompagné et que j'ai rencontré lors de cette thèse.

En premier lieu, je tiens à remercier profondément mon directeur de thèse Matthieu Fradelizi. Je l'ai connu pour la première fois lorsque j'étais en troisième année de Licence Mathématiques et Informatique à l'Université Paris-Est Marne-la-Vallée car il était un de mes enseignants. Puis, il est devenu mon tuteur pour encadrer mon mémoire de première année de Master Mathématiques et Applications. L'année suivante encore, il était mon directeur de stage de recherche de deuxième année de Master. Et aujourd'hui, Matthieu Fradelizi est mon directeur de thèse. De part ses qualités mathématiques, c'est une personne très conviviale, enjouée, motivée et toujours de bonne humeur. Ce fut un grand plaisir d'avoir travaillé avec lui pendant toutes ces années. Si ma thèse s'est bien déroulée, c'est en partie grâce à lui car il a su être disponible et à l'écoute de mes préoccupations.

D'ailleurs, le cadre même de ma thèse ne pouvait être meilleur, puisque l'Université Paris-Est Marne-la-Vallée est comme ma seconde maison, j'y est passé toute ma scolarité après Bac. Les membres du laboratoire de mathématiques de Marne sont tous très sympathiques et sont passés du statut de mes enseignants au statut de mes collègues, ce qui est une situation étrange au début, mais on s'y habitue. Je remercie par la même occasion les personnels administratifs et techniques de l'université Paris-Est Marne-la-Vallée ainsi que ceux de l'école doctorale MSTIC. Je tenais également à saluer tous les doctorants que j'ai côtoyé et avec qui j'ai passé des moments forts agréables : bon courage à vous pour la suite !

D'autre part, à travers son réseau, mon directeur de thèse m'a permis de rencontrer Sergey Bobkov, Bo'az Klartag, Grigoris Paouris, Peter Pivovarov et Artem Zvavitch. Je remercie bien évidemment toutes ces personnes pour avoir accepté de m'accueillir. Par ailleurs, je remercie grandement Bo'az Klartag qui m'a accueilli avec hospitalité à Tel-Aviv où j'y ai passé deux mois, ainsi que son étudiant Uri Grupel pour m'avoir guidé sur place. Ce fut une expérience incroyable tant mathématique que culturelle. J'ai rencontré toute une équipe dynamique et efficace dont font partie Shiri Artstein et Vitali Milman

et j'ai été surpris de voir comment un si petit pays était si avancé scientifiquement, possédant en particulier tant de mathématiciens de reconnaissance internationale.

Je remercie Sergey Bobkov et Ivan Gentil qui me font l'honneur d'être les rapporteurs de ma thèse. Je leur en suis sincèrement reconnaissant. Je tiens également à remercier François Bolley, Jean-François Bercher, Dario Cordero-Erausquin, Olivier Guédon et Mathieu Meyer d'avoir accepté de faire partie de mon jury.

Acknowledgements

I would like to express my heartfelt gratitude to the great mathematicians I met during my Ph.D.

I would like to thank Bo'az Klartag for welcoming me to Tel-Aviv where I spent two months. I would also like to thank his student Uri Grupel for showing me around Tel-Aviv. It was an amazing experience both mathematically and culturally. I met a dynamic team, which included Shiri Artstein and Vitali Milman, and I was surprised to see that such a tiny country is so advanced in science, having in particular many internationally recognized mathematicians.

I thank Sergey Bobkov, Grigoris Paouris, Peter Pivovarov and Artem Zvavitch for their invitations. I would also like to express my gratitude to Sergey Bobkov for reviewing my Ph.D. thesis.

Table des matières

Résumé	i
Travaux	v
Remerciements	ix
1 Introduction	1
1.1 Présentation de mes principaux travaux	2
1.2 Introduction	16
1.3 The Brunn-Minkowski theory	17
1.4 Functional versions	20
1.5 The Information theory	29
1.6 Relationships between the Brunn-Minkowski theory and the Information theory	46
1.7 Convex measures	51
1.8 Publications	53
2 The analogue of the concavity of entropy power in the Brunn-Minkowski theory	55
2.1 The conjecture of Costa-Cover	55
2.2 Basic properties of the parallel volume - links with geometric inequalities	57
2.3 The $\frac{1}{n}$ -concavity of the parallel volume	63
2.4 Further analogies	75
3 A functional version of the parallel volume	77
3.1 Functional version	77
3.2 Links with weighted Brascamp-Lieb-type inequalities	83

4	Concavity properties of convex measures	87
4.1	The parallel μ -volume	87
4.2	On the improvement of concavity of convex measures	97
	Bibliographie	109
	Dictionnaire	117
	Notations	123

Chapter 1

Introduction

Ma thèse est consacrée à l'étude des mesures convexes ainsi qu'à l'étude des liens entre la théorie de Brunn-Minkowski et la théorie de l'information de Shannon.

L'organisation de ma thèse est la suivante. La section 1.1 est la seule section rédigée en français et présente en détails mes travaux. Les autres sections sont rédigées exclusivement en anglais dans un souci d'universalité et de large diffusion de ma thèse. Le chapitre 1 se poursuit avec le rappel des notions de base de la théorie de Brunn-Minkowski et de la théorie de l'information, tout en mettant en lumière les liens que partagent ces deux théories. Le chapitre 2 sera consacré à mon premier article *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory* [37], en collaboration avec Matthieu Fradelizi, et publié dans *Advances in Applied Mathematics*, où nous traitons la conjecture de Costa-Cover. Lors du chapitre 3, je détaille la seconde partie de mon deuxième article *Concavity properties of extensions of the parallel volume* [64], publié dans *Mathematika*, où je démontre des propriétés de concavité pour des versions fonctionnelles de la conjecture de Costa-Cover. Enfin, au chapitre 4, j'explique la première partie de mon deuxième article [64], où il est question d'étudier les propriétés de concavité d'une version généralisée du volume parallèle, où la mesure de Lebesgue est remplacée par des mesures convexes arbitraires. Dans ce même chapitre, j'explique également mon troisième article *On the improvement of concavity of convex measures* [65], publié dans *Proceedings of the American Mathematical Society*, où il est question d'améliorer la concavité des mesures convexes sous certaines hypothèses de symétries.

En annexe, on trouve une section *Dictionnaire* et une section *Notations* où sont rappelées les terminologies essentielles ainsi que les notations utilisées dans cette thèse.

1.1 Présentation de mes principaux travaux

Ma thèse se situe dans le domaine des mathématiques et se consacre à l'étude de la géométrie des mesures convexes ainsi qu'à l'étude des liens entre la théorie de Brunn-Minkowski et la théorie de l'information de Shannon.

Les mesures convexes, définies par Borell dans les années 1970, comprennent entre autres les mesures gaussiennes, les mesures uniformément distribuées sur un ensemble convexe ou encore les mesures de Cauchy, et l'étude de la géométrie de ces mesures permet de répondre à des questions issues de l'informatique théorique.

L'inégalité de Brunn-Minkowski, qui est centrale dans la théorie de Brunn-Minkowski, permet de résoudre le célèbre problème isopérimétrique de manière simple et élégante, ce qui en fait une inégalité fondamentale en mathématiques. Le problème isopérimétrique consiste à déterminer quelle figure géométrique plane maximise l'aire lorsque le périmètre est donné. Il s'agit d'un problème d'optimisation et la solution est bien connue, c'est le cercle. L'inégalité de Brunn-Minkowski a alors attiré l'attention de nombreux mathématiciens tels que Prékopa, Leindler, Brascamp et Lieb qui en ont établi une version fonctionnelle dans les années 1970, dont la démonstration a été simplifiée dans les années 1990, grâce au développement de la méthode du transport de mesure. Récemment, des inégalités de type Brunn-Minkowski ont été obtenues sur la sphère par Cordero-Erausquin ou encore dans le cube discret par Ollivier et Villani.

Dans ma thèse, j'étudie la géométrie des mesures convexes et je démontre pour ces mesures une inégalité renforcée de type Brunn-Minkowski pour les ensembles convexes symétriques. J'établis également l'analogie de la concavité de l'entropie exponentielle dans la théorie de Brunn-Minkowski, poursuivant ainsi les travaux précurseurs de Costa, Cover, Dembo et Thomas qui ont mis en évidence, à la fin des années 1980, des liens entre la théorie de Brunn-Minkowski et la théorie de l'information de Shannon.

Avant d'explicitier en détail mes principaux résultats, commençons par situer le contexte et les motivations de mon sujet de recherche. Pour cela, faisons une brève incursion dans l'Histoire.

La théorie de Brunn-Minkowski est née des travaux précurseurs de Steiner datant de 1840 [81] ainsi que des travaux de Brunn datant de 1887 [19]. Steiner, géomètre de renom, a démontré que le volume de l'ensemble des points à distance inférieure ou égale à t d'un polyèdre (appelé *ensemble parallèle*) est une fonction polynomiale en t (voir Figure 1.1).

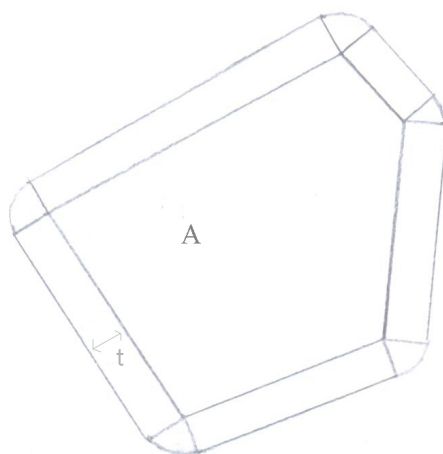


Figure 1.1: Ensemble parallèle à distance t d'un polygone A .

Brunn, durant sa thèse, a démontré que dans une direction fixée, la racine carrée de l'aire des sections d'un ensemble convexe de l'espace est une fonction concave.

Rappelons qu'un ensemble K est *convexe* si pour tous $x, y \in K$ le segment $[x, y]$ reliant x et y appartient à K . Une fonction f définie sur un ensemble convexe K est dite *convexe* si pour tous $x, y \in K$ et tout $\lambda \in [0, 1]$, l'inégalité suivante est satisfaite :

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y).$$

Enfin, une fonction f est dite *concave* si la fonction $-f$ est convexe.

Par la suite, Minkowski généralisa les travaux de Steiner et de Brunn. Dans une édition posthume de 1910 [66] dans laquelle ses travaux sont collectés, Minkowski démontre que le résultat de Steiner est valable pour tout ensemble convexe en toute dimension, et non plus seulement pour les polyèdres en dimension 3. Ceci conduisit à une importante sous-partie de la théorie de Brunn-Minkowski : la théorie des *volumes mixtes*. Minkowski généralisa également le résultat de Brunn en démontrant la formule suivante valable pour tout ensemble convexe en toute dimension, que l'on appelle aujourd'hui *inégalité de Brunn-Minkowski* :

$$|K + L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}, \quad (1.1)$$

où la somme $K + L = \{x + y ; x \in K, y \in L\}$ est appelée *somme de Minkowski* des ensembles convexes K et L , qui appartiennent à l'espace euclidien \mathbb{R}^n , et où $|\cdot|$ désigne le volume (plus précisément la mesure de Lebesgue). On déduit le résultat de Brunn en appliquant l'inégalité de Brunn-Minkowski (1.1) aux ensembles $K(t_1)$, $K(t_2)$ et $K\left(\frac{t_1+t_2}{2}\right)$

pour tous t_1, t_2 appartenant à l'ensemble des nombres réels \mathbb{R} , où, par définition, $K(t) = \{x \in K; \langle x, \theta \rangle = t\}$ désigne une section d'un ensemble convexe K dans une direction fixée θ (voir Figure 1.2).

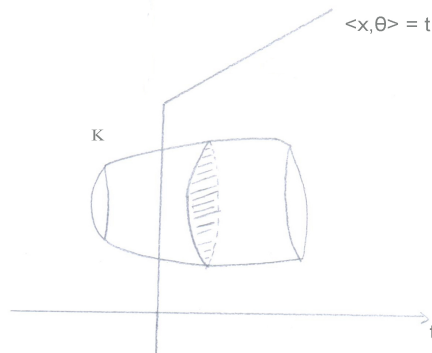


Figure 1.2: Section d'un ensemble convexe K par un hyperplan, dans la direction θ .

L'inégalité de Brunn-Minkowski a ensuite été généralisée aux ensembles compacts, non plus nécessairement convexes, par Lusternik en 1935 [63]. Cette inégalité de Brunn-Minkowski-Lusternik est d'une puissance remarquable puisqu'elle permet entre autres de démontrer l'inégalité isopérimétrique en quelques lignes. Le problème isopérimétrique, un des plus vieux problèmes mathématiques de l'humanité, demande quelle figure géométrique maximise la surface à périmètre donné, ou, de manière duale, quelle figure géométrique minimise le périmètre à surface donnée. Il est connu que la solution au problème isopérimétrique est le cercle (en dimension 2), et plus généralement la boule euclidienne en dimension quelconque. Nous débutons par définir la notion de *périmètre* d'un ensemble $A \subset \mathbb{R}^n$ de la manière suivante :

$$|\partial A|_{n-1} = \lim_{\varepsilon \rightarrow 0^+} \frac{|A + \varepsilon B_2^n| - |A|}{\varepsilon},$$

pourvu que la limite existe (ce qui sera supposé dorénavant), où B_2^n désigne la boule euclidienne unité fermée de \mathbb{R}^n .

En utilisant l'inégalité de Brunn-Minkowski (1.1), valable pour tout ensemble compact, on obtient

$$|A + \varepsilon B_2^n| \geq \left(|A|^{\frac{1}{n}} + \varepsilon |B_2^n|^{\frac{1}{n}} \right)^n \geq |A| + n\varepsilon |A|^{\frac{n-1}{n}} |B_2^n|^{\frac{1}{n}}.$$

Donc, $|\partial A|_{n-1} \geq n|A|^{\frac{n-1}{n}}|B_2^n|^{\frac{1}{n}}$. En remarquant que $|\partial B_2^n|_{n-1} = n|B_2^n|$ (identité qui s'obtient directement à partir de la définition de $|\partial B_2^n|_{n-1}$), il s'ensuit que

$$\frac{|\partial A|_{n-1}}{|A|^{1-\frac{1}{n}}} \geq \frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{1-\frac{1}{n}}}.$$

En conclusion, à volume donné, le périmètre est minimisé par la boule euclidienne. Ceci fournit une démonstration courte et élégante d'un beau résultat.

La théorie de l'information, quant à elle, a été développée suite aux travaux précurseurs de Nyquist datant de 1928 [69] ainsi qu'aux travaux de Hartley datant de la même année [47], tous deux ingénieurs en télécommunications. Le but premier d'une telle théorie était de comparer l'*efficacité* de différents systèmes de communication. Pour cela, il fallait pouvoir mesurer l'information et une telle mesure a été développée : la notion de *quantité d'information*. L'intuition d'une telle mesure est la suivante, plus un évènement a de chance de se produire, moins il apporte de l'information. On a donc considéré la quantité $-\log(p_i)$, où p_i est la probabilité d'une réalisation et où \log désigne la fonction logarithmique. D'un point de vue purement mathématiques, on aurait pu considérer n'importe quelle fonction décroissante à la place de $-\log$, mais cette fonction a été choisie à l'aide de l'expérience des ingénieurs. Suite à ces travaux précurseurs, Shannon, également ingénieur en télécommunications, développa en 1948 [78] ce que l'on appelle aujourd'hui la théorie de l'information de Shannon (voir Figure 1.3).

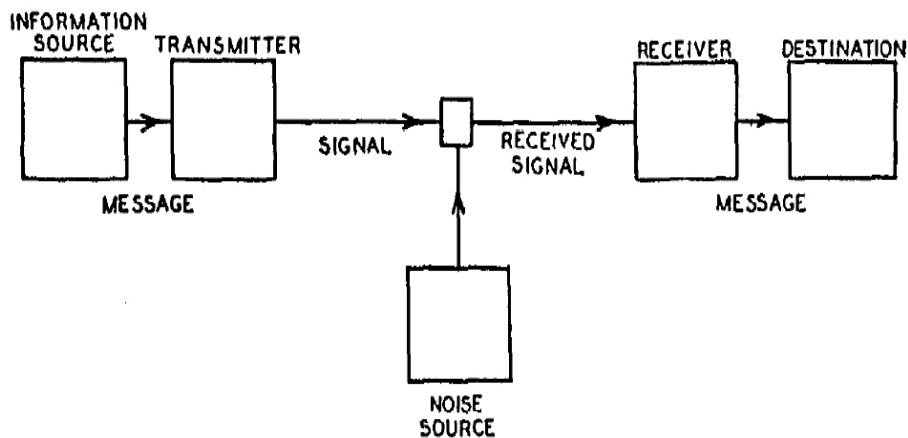


Figure 1.3: Système de communication générique, vu par Shannon [78].

Shannon développa une mesure de l'*incertitude* d'une variable aléatoire, qu'il appela

entropie. Formellement, l'entropie d'une variable aléatoire discrète X de loi donnée par $\{p_1, \dots, p_n\}$ est définie de la manière suivante :

$$h(X) = - \sum_{i=1}^n p_i \log(p_i).$$

Par exemple, si X suit une loi binomiale de paramètre $p \in [0, 1]$, alors l'entropie de X sera nulle si $p = 0$ ou $p = 1$, ce qui correspond à une variable déterministe : l'issue est connu d'avance, il n'y a aucune incertitude. L'entropie est maximale si et seulement si $p = \frac{1}{2}$, la valeur de p pour laquelle l'issue de X est la plus incertaine. Par ailleurs, parmi toutes les lois de probabilités supportées sur un ensemble fini A , c'est la loi uniforme sur A qui maximise l'entropie, ce qui concorde donc bien avec cette idée d'incertitude.

Et c'est alors que la théorie de Brunn-Minkowski et la théorie de l'information parent *communiquer* ensemble. Plusieurs mathématiciens experts en théorie de l'information ont remarqué des similitudes entre ces deux théories, notamment grâce aux travaux de Costa et Cover en 1984 [24] ainsi qu'aux travaux de Dembo, Cover et Thomas au début des années 1990 [28], [29]. Ils ont mis en lumière des analogies entre deux inégalités fondamentales à chaque théorie, l'inégalité de Brunn-Minkowski (1.1) d'une part, et l'inégalité de l'entropie exponentielle d'autre part. Cette dernière affirme que pour toutes variables aléatoires X et Y à densité et indépendantes, l'inégalité suivante est vérifiée :

$$N(X + Y) \geq N(X) + N(Y), \tag{1.2}$$

où $N(X) = \frac{1}{2\pi e} \exp\left(\frac{2}{n}H(X)\right)$, $H(X)$ représentant l'entropie d'une variable aléatoire continue X de densité f , définie par :

$$H(X) = - \int_{\mathbb{R}^n} f(x) \log(f(x)) dx,$$

pourvu que l'intégrale existe.

Au-delà de ces ressemblances purement formelles, il existe une approche qui fait le pont entre la théorie de Brunn-Minkowski et la théorie de l'information, notamment en unifiant l'inégalité de Brunn-Minkowski (1.1) et l'inégalité de l'entropie exponentielle (1.2). Pour cela, nous avons besoin d'une quantité issue de la théorie de l'information, appelée la *p-ième entropie de Rényi*. Pour $p \in [0, +\infty]$, on définit la

p -ième entropie de Rényi d'une variable aléatoire X de densité f par

$$H_p(X) = \frac{1}{1-p} \log \left(\int_{\mathbb{R}^n} f^p(x) dx \right),$$

où les cas limites sont interprétés par continuité. Notamment, on obtient

$$\lim_{p \rightarrow 1} H_p(X) = H(X), \quad \lim_{p \rightarrow 0} H_p(X) = \log(|\text{supp}(f)|),$$

où $|\text{supp}(f)|$ désigne le volume du support de f . On comprend donc que l'existence d'une inégalité entropique pour la p -ième entropie de Rényi permettrait de déduire à la fois l'inégalité de Brunn-Minkowski (1.1) et l'inégalité de l'entropie exponentielle (1.2) en passant à la limite lorsque p tend vers 0 et 1. Une telle inégalité existe bien et découle d'une inégalité de Beckner, Brascamp et Lieb prouvée au milieu des années 1970 [6], [17], et qui n'est rien d'autre que l'inégalité de convolution de Young avec constante exacte.

Ainsi, on est face à deux théories développées durant des époques différentes, dans un contexte différent et des préoccupations différentes, d'une part par des géomètres et d'autre part par des ingénieurs, et finalement on constate que ces deux théories sont très proches et possèdent des racines communes. Le constat de pouvoir faire le pont entre des théories très différentes en apparence fait des mathématiques une science séduisante.

Je vais maintenant présenter l'essentiel de mes travaux. Dans une première partie de ma thèse, je me consacre à l'étude des liens entre la théorie de Brunn-Minkowski et la théorie de l'information en étudiant certaines conjectures proposées par Costa, Cover, Dembo et Thomas. Tout particulièrement, Costa [23] a démontré que l'entropie exponentielle d'une variable aléatoire X à laquelle on ajoute un bruit gaussien de variance t indépendant de X (autrement dit $N(X + \sqrt{t}Z)$, où Z représente une variable aléatoire de loi gaussienne indépendante de X) est concave sur \mathbb{R}_+ en tant que fonction de t (Villani [84] a ensuite découvert une preuve très courte de ce résultat). Comme analogue, Costa et Cover [24] ont alors conjecturé que la racine n -ième du volume parallèle d'un ensemble compact A à distance t (autrement dit $|A + tB_2^n|^{\frac{1}{n}}$, où B_2^n représente la boule euclidienne unité de \mathbb{R}^n) est concave sur \mathbb{R}_+ en tant que fonction de t . Dans ma thèse, je résous complètement cette conjecture en démontrant le résultat suivant :

Théorème 1. [37] Soit A un ensemble compact de \mathbb{R}^n . On note, pour $t \in \mathbb{R}_+$, $V_A(t) = |A + tB_2^n|$ le volume parallèle de A à distance t . Alors,

1. En dimension 1, V_A est concave sur \mathbb{R}_+ .

2. En dimension 2, si A est connexe, alors $V_A^{\frac{1}{2}}$ est concave sur \mathbb{R}_+ . De plus, il existe des ensembles non connexes A tels que V_A ne soit pas concave sur \mathbb{R}_+ .
3. En dimension $n \geq 3$, il existe des ensembles étoilés A tels que $V_A^{\frac{1}{n}}$ ne soit pas concave sur \mathbb{R}_+ .
4. Si la fonction $\varepsilon \mapsto |\varepsilon A + B_2^n|$ est deux fois dérivable dans un voisinage de 0 de dérivée seconde continue en 0, alors il existe $t_0 \geq 0$ tel que $V_A^{\frac{1}{n}}$ est concave sur $[t_0, +\infty)$. En particulier, ceci est valable lorsque A est un ensemble fini.

Dans le cas de la dimension 1 (premier point du théorème 1), j'ai pu calculer explicitement la fonction V_A et donc résoudre le problème directement par calculs. Ceci n'est plus vrai en dimension supérieure.

Dans le cas de la dimension 2 (second point du théorème 1), la preuve s'effectue en approximant l'ensemble A par un ensemble fini de points. J'ai alors été amené à étudier ce que l'on appelle les *cellules de Dirichlet-Voronoi*. Lorsque l'ensemble A est fini, la fonction V_A est deux fois dérivable en dehors d'un nombre fini de points, et une analyse précise des dérivées première et seconde m'a permis de démontrer le résultat voulu.

En toute généralité, la conjecture de Costa-Cover est fautive, et mon idée pour construire un contre-exemple a été de considérer la réunion d'un ensemble convexe et d'un espace de dimension $n - 2$ (voir Figure 1.4).

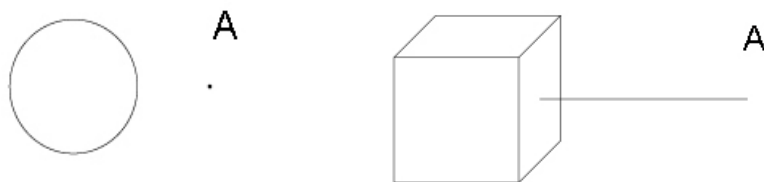


Figure 1.4: Contre-exemple en dimension 2 (à gauche) ; Contre-exemple en dimension 3 (à droite).

Précisons que l'on peut démontrer que la conjecture de Costa-Cover est valide lorsque l'ensemble A est convexe, à l'aide de l'inégalité de Brunn-Minkowski (1.1). En effet, si

l'ensemble A est convexe, alors pour tout $\lambda \in [0, 1]$ on a $A = (1 - \lambda)A + \lambda A$. On obtient donc que pour tous nombres positifs $t_1, t_2 \in \mathbb{R}_+$,

$$\begin{aligned} |A + ((1 - \lambda)t_1 + \lambda t_2)B_2^n|^{\frac{1}{n}} &= |(1 - \lambda)(A + t_1 B_2^n) + \lambda(A + t_2 B_2^n)|^{\frac{1}{n}} \\ &\geq (1 - \lambda)|A + t_1 B_2^n|^{\frac{1}{n}} + \lambda|A + t_2 B_2^n|^{\frac{1}{n}}. \end{aligned}$$

Je mets également en évidence des liens entre la conjecture de Costa-Cover et certaines inégalités géométriques fondamentales telles que l'inégalité isopérimétrique et l'inégalité de Brunn-Minkowski. Par exemple, leur conjecture implique que pour tout $t \geq 0$ et tout ensemble A tel que $(V_A^{1/n})'_+(0)$ existe,

$$\frac{1}{n} \frac{|\partial A|_{n-1}}{|A|^{1-\frac{1}{n}}} = (V_A^{1/n})'_+(0) \geq (V_A^{1/n})'_+(t) \geq \lim_{t \rightarrow +\infty} (V_A^{1/n})'_+(t) = \frac{1}{n} \frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{1-\frac{1}{n}}},$$

où $(V_A^{1/n})'_+(t)$ désigne la dérivée à droite de $V_A^{1/n}$ au point t , qui existe toujours lorsque $t > 0$. On obtient donc un résultat plus fort que l'inégalité isopérimétrique : on obtient un chemin décroissant de $\frac{|\partial A|_{n-1}}{|A|^{1-\frac{1}{n}}}$ à $\frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{1-\frac{1}{n}}}$ à travers la famille

$$\left(\frac{|\partial(A + tB_2^n)|_{n-1}}{|A + tB_2^n|^{1-\frac{1}{n}}} \right)_{t \in \mathbb{R}_+}.$$

J'étudie également la validité de la conjecture de Costa-Cover pour des valeurs de t suffisamment grandes et démontre un tel résultat sous certaines conditions de régularité du volume parallèle (dernier point du théorème 1).

Récemment, Savaré et Toscani [75] ont démontré que la p -ième entropie exponentielle de Rényi est concave, et cela pour tout $p > 1 - \frac{2}{n}$, sans préciser ce qu'il se passait pour $p \leq 1 - \frac{2}{n}$. Le théorème 1 montre qu'une telle concavité n'est pas valable lorsque p est dans un voisinage de 0.

En géométrie convexe, il est courant d'étendre des inégalités à un contexte non géométrique, ceci permettant parfois de démontrer plus facilement des résultats purement géométriques, et nouant ainsi de nouveaux liens avec d'autres théories mathématiques. Notamment, l'inégalité de Brunn-Minkowski peut se généraliser de plusieurs manières, nous en avons déjà vu une précédemment en considérant la p -ième entropie de Rényi. Une autre méthode de généralisation consiste à développer des versions fonctionnelles. Par exemple, l'inégalité de Prékopa-Leindler [70], [60], [71] prouvée au début des années 1970, établit une version fonctionnelle de l'inégalité de Brunn-Minkowski.

J'ai développé une version fonctionnelle de la conjecture de Costa-Cover en considérant la fonction $h_t(z) = \sup_{z=x+ty} f(x)g(y)^t$, $t \in \mathbb{R}_+$, où f, g sont deux fonctions mesurables positives définies sur \mathbb{R}^n . En prenant des fonctions indicatrices $f = 1_A$ et $g = 1_{B_2^n}$, où A est un ensemble compact de \mathbb{R}^n , on obtient que $h = 1_{A+tB_2^n}$. Dans ma thèse, j'ai considéré et étudié une généralisation de cette version fonctionnelle et je démontre le résultat suivant :

Théorème 2. [64]

1. Soit $\gamma \geq -\frac{1}{n}$. Soient $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ deux fonctions γ -concaves (une fonction f est γ -concave si γf^γ est concave). Alors la fonction $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$ est $\frac{\gamma}{1+\gamma n}$ -concave sur \mathbb{R}_+ , où

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; g(y)>0}} (f(x)^\gamma + tg(y)^\gamma)^{\frac{1}{\gamma}}, \quad h_t^{(0)}(z) = \sup_{z=x+ty} f(x)g(y)^t.$$

2. Soit $\gamma \leq 0$. Soit $f : \mathbb{R} \rightarrow \mathbb{R}_+$ telle que f^γ (à interpréter par $-\log(f)$ lorsque $\gamma = 0$) soit une fonction positive, lipschitzienne et bornée. Définissons pour tout $y \in \mathbb{R}$, $V(y) = \frac{|y|^p}{p}$ avec $p \geq 1$, et

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; V(y)>0}} (f(x)^\gamma + tV(y))^{\frac{1}{\gamma}}, \quad h_t^{(0)}(z) = \sup_{z=x+ty} f(x)e^{-tV(y)}.$$

Si $h_t^{(\gamma)}$ est de classe C^2 et si $\lim_{z \rightarrow \pm\infty} \frac{\partial}{\partial z} h_t^{(\gamma)}(z) = 0$, alors la fonction $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz$ est concave sur \mathbb{R}_+ .

Le premier point du théorème 2 repose sur l'inégalité de Borell, Brascamp et Lieb [15], [18], qui est une inégalité plus précise que l'inégalité de Prékopa-Leindler.

Pour le second point du théorème 2, la fonction $V = g^\gamma$ est convexe lorsque $\gamma < 0$, et j'ai alors relié la fonction $h_t^{(\gamma)}$ avec la solution de Hopf-Lax de l'équation de Hamilton-Jacobi de la manière suivante :

$$h_t^{(\gamma)}(z) = \sup_{x \in \mathbb{R}^n} \left(f(x)^\gamma + tV \left(\frac{z-x}{t} \right) \right)^{\frac{1}{\gamma}} = \left(Q_t^{(V)} f^\gamma(z) \right)^{\frac{1}{\gamma}},$$

où pour une fonction convexe arbitraire V et une fonction arbitraire u ,

$$Q_t^{(V)}u(z) = \inf_{x \in \mathbb{R}^n} \left(u(x) + tV \left(\frac{z-x}{t} \right) \right).$$

Il est connu que $Q_t^{(V)}u$ est la solution de Hopf-Lax de l'équation de Hamilton-Jacobi, et une analyse précise de cette solution m'a permis d'obtenir le second point du théorème 2.

Voyons maintenant encore une autre manière de généraliser l'inégalité de Brunn-Minkowski, cette fois-ci en généralisant la mesure de Lebesgue. Borell [15] a étudié en 1975 ce que l'on appelle aujourd'hui les *mesures s -concaves*, où s est un paramètre dans $[-\infty, +\infty]$. Ces mesures vérifient par définition l'inégalité de type Brunn-Minkowski suivante :

$$\mu((1-\lambda)A + \lambda B) \geq ((1-\lambda)\mu(A)^s + \lambda\mu(B)^s)^{\frac{1}{s}}, \quad (1.3)$$

pour tout $\lambda \in [0, 1]$ et tous compacts $A, B \subset \mathbb{R}^n$ tels que $\mu(A)\mu(B) > 0$. La mesure de Lebesgue apparaît alors comme étant une mesure $\frac{1}{n}$ -concave. La partie droite de l'inégalité (1.3) doit être interprétée par continuité, c'est-à-dire par $\mu(A)^{1-\lambda}\mu(B)^\lambda$ lorsque $s = 0$, par $\min(\mu(A), \mu(B))$ lorsque $s = -\infty$ et par $\max(\mu(A), \mu(B))$ lorsque $s = +\infty$. Le cas $s = 0$ correspond à ce que l'on appelle *mesures log-concaves*, une classe de mesures dans laquelle la mesure gaussienne, fondamentale en probabilités, appartient. En conséquence de l'inégalité de Hölder, toute mesure s -concave est r -concave pour tout $r \leq s$. En particulier, toute mesure s -concave est $-\infty$ -concave et ces dernières sont appelées *mesures convexes*.

Maintenant que nous avons introduit les mesures convexes, il est assez aisé d'imaginer une autre extension de la conjecture de Costa-Cover. Il suffit de remplacer la mesure de Lebesgue par une mesure s -concave. Dans ma thèse, je démontre le résultat suivant :

Théorème 3. [64] Soit $s \in [-\infty, \frac{1}{2}] \cup [1, +\infty]$. Soient A un ensemble compact de \mathbb{R} et μ une mesure s -concave de \mathbb{R} . Alors, la fonction $t \mapsto V_A^\mu(t) = \mu(A + tB_{\frac{1}{2}})$ est s -concave sur \mathbb{R}_+ . De plus, il existe des mesures s -concaves, avec $s \in]\frac{1}{2}, 1[$, tels que V_A^μ ne soit pas s -concave sur \mathbb{R}_+ .

Je démontre ce résultat en utilisant une technique de localisation développée par Kannan, Lovász et Simonovits [62], [56] dans les années 1990, sous une autre forme établie par Fradelizi et Guédon [35] en 2004.

Par ailleurs, la conclusion du théorème 3 reste valable en toute dimension lorsque $s > 1$, puisque dans ce cas, la seule mesure s -concave non triviale est la mesure de Dirac en un point. La conclusion reste également valable en toute dimension lorsque $s = -\infty$ car la fonction $t \mapsto V_A^\mu(t)$ est croissante, et en toute dimension pour toute valeur de s lorsque l'ensemble A est convexe. Cependant, puisque la conjecture de Costa-Cover est fautive en toute généralité en dimension 2, on ne peut pas s'attendre à obtenir les résultats du théorème 3 en dimension supérieure ou égale à 2. Il en est de même pour le théorème 2.

La dernière partie de ma thèse se consacre à l'étude de l'amélioration de la concavité des mesures convexes sous certaines hypothèses telles que la symétrie. Précisément, je mets en avant le problème de déterminer quelles sont les mesures qui satisfont une concavité au sens de Borell en fonction de la classe d'ensembles considérée :

Problème 1. Pour quelle valeur s dans $[-\infty, +\infty]$, pour quelle classe \mathcal{M} de mesures positives et pour quelle classe \mathcal{C} de couples d'ensembles boréliens de \mathbb{R}^n a-t-on

$$\mu((1 - \lambda)A + \lambda B) \geq ((1 - \lambda)\mu(A)^s + \lambda\mu(B)^s)^{\frac{1}{s}} \quad (1.4)$$

pour tout $\mu \in \mathcal{M}$, pour tout $(A, B) \in \mathcal{C}$ tels que $\mu(A)\mu(B) > 0$ et pour tout $\lambda \in [0, 1]$?

Dans le cas où aucune restriction n'est supposée sur la classe \mathcal{C} , le problème 1 a été étudié par Borell [15]. Dans le cas où seule la mesure de Lebesgue est considérée, le problème 1 a été étudié par Bonnesen [13], et est toujours étudié (voir par exemple [49]).

Notons que le problème 1 admet une solution si $s = +\infty$, si \mathcal{M} est la classe des mesures convexes et si \mathcal{C} est la classe des couples d'ensembles de même mesure. En effet, on a alors pour tout $\lambda \in [0, 1]$,

$$\mu((1 - \lambda)A + \lambda B) \geq \min(\mu(A), \mu(B)),$$

par définition. Puisque $\mu(A) = \mu(B)$, il s'ensuit que

$$\mu((1 - \lambda)A + \lambda B) \geq \mu(A) = \max(\mu(A), \mu(B)).$$

Ce problème fait suite à des travaux très récents de Gardner et Zvavitch [41] dans lesquels les auteurs démontrent que la mesure gaussienne satisfait l'inégalité de Brunn-

Minkowski pour une classe d'ensembles très spécifique. Leur étude se restreint donc au cas de la mesure gaussienne et je généralise leurs résultats à des mesures plus générales ainsi qu'à une classe d'ensembles plus générale. Avant d'énoncer mes résultats obtenus, précisons qu'une fonction f est *inconditionnelle* si pour tout $(x_1, \dots, x_n) \in \mathbb{R}^n$ et tout $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, on a $f(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) = f(x_1, \dots, x_n)$. De plus, une mesure à densité est inconditionnelle si sa densité est inconditionnelle, ce qui est le cas de la mesure de Lebesgue ou de la mesure gaussienne, par exemple. Enfin, un ensemble est inconditionnel si sa fonction indicatrice est inconditionnelle, ce qui est le cas du cube ou de la boule euclidienne, par exemple.

Théorème 4. [65]

1. Soit μ une mesure log-concave inconditionnelle et soit $A \subset \mathbb{R}^n$ un ensemble convexe inconditionnel. Alors, pour tous $A_1, A_2 \in \{\alpha A; \alpha > 0\}$ et pour tout $\lambda \in [0, 1]$,

$$\mu((1 - \lambda)A_1 + \lambda A_2)^{\frac{1}{n}} \geq (1 - \lambda)\mu(A_1)^{\frac{1}{n}} + \lambda\mu(A_2)^{\frac{1}{n}}.$$

2. Soient $\mu_i, 1 \leq i \leq n$, des mesures de densités $\phi_i : \mathbb{R} \rightarrow \mathbb{R}_+$ telles que les ϕ_i soient croissantes sur $(-\infty, 0]$ et décroissantes sur $[0, +\infty)$. Soit μ la mesure produit de μ_1, \dots, μ_n et soient $A, B \subset \mathbb{R}^n$ le produit de n sous-ensembles mesurables de \mathbb{R} tels que $0 \in A \cap B$. Alors, pour tout $\lambda \in [0, 1]$,

$$\mu((1 - \lambda)A + \lambda B)^{\frac{1}{n}} \geq (1 - \lambda)\mu(A)^{\frac{1}{n}} + \lambda\mu(B)^{\frac{1}{n}}.$$

3. Soit μ_1 une mesure de densité $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ telle que ϕ soit croissante sur $(-\infty, 0]$ et décroissante sur $[0, +\infty)$. Soit μ_2 une mesure $(n - 1)$ -dimensionnelle et soit μ la mesure produit de μ_1 et μ_2 . Soit $A = A_1 \times \mathbb{R}^{n-1}$, où A_1 est un sous-ensemble mesurable de \mathbb{R} et soit B un sous-ensemble mesurable de \mathbb{R}^n tels que $0 \in A \cap B$. Alors, pour tout $\lambda \in [0, 1]$,

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

4. Soit μ une mesure de densité $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ telle que ϕ soit $-\frac{1}{n-1}$ -concave et soient A, B des sous-ensembles mesurables de \mathbb{R}^n . Supposons qu'il existe u un vecteur de

norme 1 tel que

$$\sup_{t \in \mathbb{R}} \mu_{n-1}(A \cap (u^\perp + tu)) = \sup_{t \in \mathbb{R}} \mu_{n-1}(B \cap (u^\perp + tu)),$$

où $\mu_{n-1}(A \cap (u^\perp + tu)) = \int_{A \cap (u^\perp + tu)} \phi(x) dx$ et de même pour $\mu_{n-1}(B \cap (u^\perp + tu))$, alors pour tout $\lambda \in [0, 1]$,

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

Pour démontrer le premier point du théorème 4, j'utilise un résultat de Cordero-Erausquin, Fradelizi et Maurey [21], appelé (B)-théorème, qui affirme que la fonction $t \mapsto \mu(e^t A)$ est concave sur \mathbb{R} pour toute mesure μ log-concave inconditionnelle et tout ensemble A convexe inconditionnel. Précisément, je démontre que cela implique nécessairement que la fonction $t \mapsto \mu(tA)^{\frac{1}{n}}$ est concave sur \mathbb{R}_+ . D'où je déduis le premier point du théorème 4.

Concernant le second point, j'applique un résultat de Henstock et Macbeath [48] établissant une amélioration de l'inégalité de Prékopa-Leindler en dimension 1. Puis, j'utilise une propriété de tensorisation pour déduire le résultat en dimension quelconque.

Je déduis le troisième point du théorème 4 à partir du second point en remarquant que si l'un des ensembles est de la forme $A_1 \times \mathbb{R}^{n-1}$, où A_1 est un sous-ensemble de \mathbb{R} , alors le problème se réduit à un problème en dimension 1. J'applique alors le second point en prenant $n = 1$.

J'obtiens le dernier point à l'aide d'une version fonctionnelle d'un résultat célèbre de Bonnesen [13] qui affirme que si deux ensembles A et B dans \mathbb{R}^n partagent une même section maximale, alors pour tout $\lambda \in [0, 1]$,

$$|(1 - \lambda)A + \lambda B| \geq (1 - \lambda)|A| + \lambda|B|.$$

L'intérêt de développer des inégalités de type Brunn-Minkowski pour des mesures arbitraires réside essentiellement dans le fait que l'on peut obtenir de nouvelles inégalités isopérimétriques. Il s'agit d'un problème difficile et une complète caractérisation de l'inégalité isopérimétrique n'est actuellement connue que pour la mesure de Lebesgue (qui est fondamentale en géométrie) et la mesure gaussienne (qui est fondamentale en probabilités).

Pour conclure, j'ai été confronté durant ma thèse à d'innombrables problèmes liés aux domaines de la théorie de Brunn-Minkowski et de la théorie de l'information. Il s'agit de domaines de recherches très actifs auxquels j'ai le plaisir d'avoir pu apporter ma pierre à l'édifice, et dans lesquels j'ai trouvé tant d'intéressantes questions ouvertes et de beaux problèmes à étudier.

1.2 Introduction

In my thesis I pursue the study of convex measures as well as the relationships between the Brunn-Minkowski theory and the Information theory. My thesis is situated in the fields of Convex Geometry, the Brunn-Minkowski theory, and the Information theory.

The organization of the thesis is as follows. In this first chapter I start with a short historical introduction to the Brunn-Minkowski theory and the Information theory. In particular, I explain the fundamental tools that arise in Convex and non-Convex Geometry (such as functional forms of geometric inequalities), the analogies between the Brunn-Minkowski theory and the Information theory, and the convex measures. The second chapter is devoted to my first paper *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory* [37], coauthored with Matthieu Fradelizi, published in *Advances in Applied Mathematics*. In the third chapter, I explain the second part of my second paper *Concavity properties of extensions of the parallel volume* [64], published in *Mathematika*. Finally, in the last chapter, I explain the first part of my second paper [64] and also my third paper *On the improvement of concavity of convex measures* [65], published in *Proceedings of the American Mathematical Society*.

In appendix, one will find a section *Dictionary* and a section *Notations* that recall the basic terminology and notations used in this thesis.

1.3 The Brunn-Minkowski theory

The Brunn-Minkowski theory was founded mainly through the works of Steiner, Brunn and Minkowski. In his Ph.D. thesis, Brunn noticed that the volume of the sections of a convex body, in a fixed direction, has a concavity property. More precisely, he showed that the function $t \mapsto |\{x \in K; \langle x, \theta \rangle = t\}|_2$ is a $\frac{1}{2}$ -concave function on \mathbb{R} , where $|\cdot|_2$ denotes the Lebesgue measure on the plane, where K is a convex body in \mathbb{R}^3 and $\theta \in \mathcal{S}^{n-1}$ is a direction (see Figure (1.5) below).

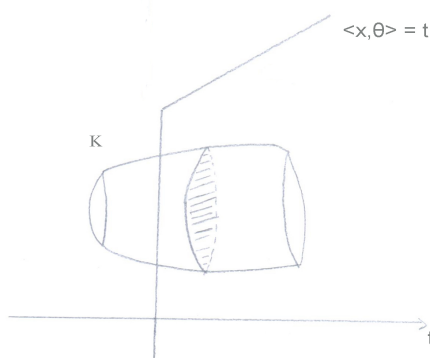


Figure 1.5: Hyperplane section of a convex body

Similar to Brunn's contemporaries, many of his works were stated in dimension 2 or 3, although most of his results can be generalized in higher dimension.

Soon after, Minkowski highlighted and generalized the result of Brunn. His achievement in mathematics is impressive and his works are collected in a posthumous edition [66]. Minkowski proved the following result, nowadays called the *Brunn-Minkowski inequality*:

Theorem 1.3.1 (Minkowski [66]). *Let K and L be two convex bodies in \mathbb{R}^n , then*

$$|K + L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}. \quad (1.5)$$

Here, $|\cdot|$ denotes the n -dimension Lebesgue measure.

Later, Lusternik in [63] extended the Brunn-Minkowski inequality (1.5) to arbitrary measurable sets in \mathbb{R}^n such that the sum is measurable. Therefore, Lusternik definitively completed the initial result of Brunn (staying in Geometry). Nowadays, one sometimes refers to the Brunn-Minkowski-Lusternik inequality for the following inequality:

Theorem 1.3.2 (Lusternik [63]). *Let A and B be two compact subsets of \mathbb{R}^n , then*

$$|A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}. \quad (1.6)$$

The name of Lusternik is often omitted.

Replacing A with $(1 - \lambda)A$ and B with λB for $\lambda \in [0, 1]$, one obtains the following equivalent formulation

$$|(1 - \lambda)A + \lambda B|^{\frac{1}{n}} \geq (1 - \lambda)|A|^{\frac{1}{n}} + \lambda|B|^{\frac{1}{n}}. \quad (1.7)$$

The Brunn-Minkowski inequality (1.7) is a fundamental inequality in Mathematics. The most important reason is that this inequality solves the (classical) isoperimetric inequality in a few lines. The isoperimetric inequality is a problem coming from real preoccupations, namely it asks which geometric object minimizes the perimeter at a given volume. The answer is Euclidean balls (circles in dimension 2, spheres in dimension 3). Moreover, this proof of the isoperimetric inequality works in every dimension.

Let us see how the isoperimetric inequality results from the Brunn-Minkowski inequality (1.6). Let us define the *outer Minkowski surface area* by

$$|\partial A| = \lim_{\varepsilon \rightarrow 0^+} \frac{|A + \varepsilon B_2^n| - |A|}{\varepsilon},$$

whenever the limit exists.

Corollary 1.3.3 (Isoperimetric inequality). *Let A be a compact subset of \mathbb{R}^n . Then,*

$$\frac{|\partial A|}{|A|^{1-\frac{1}{n}}} \geq \frac{|\partial B_2^n|}{|B_2^n|^{1-\frac{1}{n}}}.$$

Proof. Using Brunn-Minkowski inequality (1.6), one has

$$|A + \varepsilon B_2^n| \geq \left(|A|^{\frac{1}{n}} + \varepsilon |B_2^n|^{\frac{1}{n}} \right)^n \geq |A| + n\varepsilon |A|^{\frac{n-1}{n}} |B_2^n|^{\frac{1}{n}}.$$

Thus, $|\partial A| \geq n|A|^{\frac{n-1}{n}} |B_2^n|^{\frac{1}{n}}$. Notice that $|\partial B_2^n| = n|B_2^n|^{\frac{1}{n}}$. Hence,

$$\frac{|\partial A|}{|A|^{1-\frac{1}{n}}} \geq \frac{|\partial B_2^n|}{|B_2^n|^{1-\frac{1}{n}}}.$$

□

Another important part of the Brunn-Minkowski theory is the notion of *mixed volume*. This notion has been highlighted by the precursor works of Steiner in [81] who discovered that the *parallel volume* of special convex bodies has a polynomial expansion.

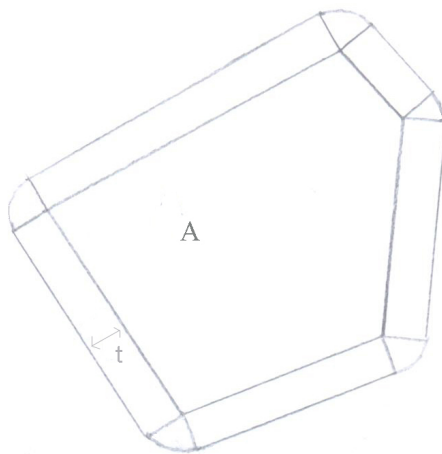


Figure 1.6: Parallel volume of a polygon

Theorem 1.3.4 (Steiner [81]). *Let A be a convex polyhedron of \mathbb{R}^3 , then*

$$|A + tB_2^3| = a_0 + a_1t + a_2t^2 + a_3t^3,$$

where a_0, a_1, a_2, a_3 are non-negative quantities depending on A and B_2^3 .

In the example of Figure 1.6 above where A is a polygon, one can see that

$$|A + tB_2^2| = |A| + P(A)t + \pi t^2,$$

where $P(A)$ is the perimeter of A ; and one can see that this result holds for every polygon.

Nowadays, one refers to the *Steiner formula* for the following identity:

Theorem 1.3.5 (Steiner's formula). *Let A be a compact convex subset of \mathbb{R}^n , then for every $t \geq 0$,*

$$|A + tB_2^n| = \sum_{i=0}^n \binom{n}{i} V(A, [i]; B_2^n, [n-i])t^{n-i},$$

where $V(A, [i]; B_2^n, [n-i])$ are called the *mixed volumes* of A and B_2^n .

1.4 Functional versions

One can link the field of Geometry to the field of Analysis by applying functional inequalities to particular functions: indicator of sets. We will see how to derive the Brunn-Minkowski inequality from a functional inequality: the Borell-Brascamp-Lieb inequality.

Theorem 1.4.1 (Borell [15], Brascamp-Lieb [18]). *Let $\gamma \geq -\frac{1}{n}$. Let $\lambda = (\lambda_1, \dots, \lambda_m)$, where $\lambda_1, \dots, \lambda_m$ are positive numbers such that $\sum_{i=1}^m \lambda_i = 1$, and let $f_1, \dots, f_m, h : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be non-negative measurable functions satisfying for every $x_1, \dots, x_m \in \mathbb{R}^n$ such that $f_1(x_1) \cdots f_m(x_m) > 0$,*

$$h(\lambda_1 x_1 + \cdots + \lambda_m x_m) \geq M_\lambda^\gamma(f_1(x_1), \dots, f_m(x_m)).$$

Then,

$$\int_{\mathbb{R}^n} h(x) \, dx \geq M_\lambda^{\frac{\gamma}{1+\gamma n}} \left(\int_{\mathbb{R}^n} f_1(x) \, dx, \dots, \int_{\mathbb{R}^n} f_m(x) \, dx \right).$$

Here,

$$M_\lambda^\theta(a_1, \dots, a_m) = \begin{cases} (\lambda_1 a_1^\theta + \cdots + \lambda_m a_m^\theta)^{\frac{1}{\theta}} & \text{if } \theta \in \mathbb{R} \setminus \{0\} \\ a_1^{\lambda_1} \cdots a_m^{\lambda_m} & \text{if } \theta = 0 \\ \min(a_1, \dots, a_m) & \text{if } \theta = -\infty \\ \max(a_1, \dots, a_m) & \text{if } \theta = +\infty \end{cases}.$$

In Theorem 1.4.1, the case $\gamma > 0$ in dimension 1 has been proved by Henstock and Macbeath in [48] and the case $\gamma = 0$ has been proved by Prékopa and Leindler in [70], [60], [71].

Let us rewrite the case $\gamma = 0$ of Theorem 1.4.1 for two functions.

Theorem 1.4.2 (Prékopa-Leindler's inequality). *Let f and g be two non-negative measurable functions defined on \mathbb{R}^n . For every $\lambda \in [0, 1]$, one has*

$$\int_{\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)g(y) \, dz \geq \left(\int_{\mathbb{R}^n} f(x)^{\frac{1}{1-\lambda}} \, dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)^{\frac{1}{\lambda}} \, dy \right)^\lambda. \quad (1.8)$$

A few years later, Beckner, Brascamp and Lieb gave another generalization of the Prékopa-Leindler inequality (1.8).

Theorem 1.4.3 (Beckner [6], Brascamp-Lieb [17]). *Let $0 \leq p, q, r \leq +\infty$ such that $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ and $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$. Let us denote $c_p = \frac{p^{\frac{1}{p}}}{|p'|^{\frac{1}{p'}}}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and similarly c_q, c_r .*

If $1 \leq p, q, r \leq +\infty$, then

$$\sup_{f \in L^p, g \in L^q} \frac{\|f \star g\|_{L^r(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}} = \left(\frac{c_p c_q}{c_r} \right)^{\frac{n}{2}}. \quad (1.9)$$

If $0 \leq p, q, r \leq 1$, then

$$\inf_{f \in L^p, g \in L^q} \frac{\|f \star g\|_{L^r(\mathbb{R}^n)}}{\|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}} = \left(\frac{c_p c_q}{c_r} \right)^{\frac{n}{2}}. \quad (1.10)$$

The case $1 \leq p, q, r \leq +\infty$ in Theorem 1.4.3 has been shown simultaneously by Beckner [6] and Brascamp, Lieb [17]. However, contrary to Brascamp and Lieb, Beckner did not examine the case $0 \leq p, q, r \leq 1$.

One derives the Prékopa-Leindler inequality (1.8) from inequality (1.10) by considering

$$h_R(z) = \left(\int_{\mathbb{R}^n} f_{1-\lambda}(z-y)^{(1-\lambda)R} g_\lambda(y)^{\lambda R} dy \right)^{\frac{1}{R-1}},$$

where $f_{1-\lambda}(x) = f\left(\frac{x}{1-\lambda}\right)$ and $g_\lambda(x) = g\left(\frac{x}{\lambda}\right)$. By approximation, one may assume that f and g are bounded and compactly supported. One then has

$$\begin{aligned} \lim_{R \rightarrow +\infty} h_R(z) &= \sup_{y \in \mathbb{R}^n} f\left(\frac{z-y}{1-\lambda}\right)^{1-\lambda} g\left(\frac{y}{\lambda}\right)^\lambda \\ &= \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda \\ &:= h(z), \end{aligned}$$

and furthermore, for $R > \max\left(\frac{1}{1-\lambda}, \frac{1}{\lambda}\right)$,

$$\begin{aligned} \|h_R\|_{L^1(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f_{1-\lambda}(x-y)^{(1-\lambda)R} g_\lambda(y)^{\lambda R} dy \right)^{\frac{1}{R-1}} dx \\ &= \|f_{1-\lambda}^{(1-\lambda)R} \star g_\lambda^{\lambda R}\|_{L^{\frac{1}{R-1}}(\mathbb{R}^n)} \\ &\geq \left(\left(\frac{c_p c_q}{c_r} \right)^{\frac{n}{2}} \|f_{1-\lambda}^{(1-\lambda)R}\|_{L^p(\mathbb{R}^n)} \|g_\lambda^{\lambda R}\|_{L^q(\mathbb{R}^n)} \right)^{\frac{1}{R-1}}, \end{aligned}$$

where $\frac{1}{p} = (1 - \lambda)R$, $\frac{1}{q} = \lambda R$. Notice that

$$\begin{aligned} \|f_{1-\lambda}^{(1-\lambda)R}\|_{L^p(\mathbb{R}^n)} &= \left((1-\lambda)^n \|f\|_{L^1(\mathbb{R}^n)} \right)^{(1-\lambda)R}, \\ \|g_\lambda^{\lambda R}\|_{L^q(\mathbb{R}^n)} &= \left(\lambda^n \|g\|_{L^1(\mathbb{R}^n)} \right)^{\lambda R}. \end{aligned}$$

Thus,

$$\|h_R\|_{L^1(\mathbb{R}^n)} \geq \left(\frac{c_p c_q}{c_r} \right)^{\frac{n}{2} \frac{1}{R-1}} \left((1-\lambda)^n \|f\|_{L^1(\mathbb{R}^n)} \right)^{(1-\lambda) \frac{R}{R-1}} \left(\lambda^n \|g\|_{L^1(\mathbb{R}^n)} \right)^{\lambda \frac{R}{R-1}}. \quad (1.11)$$

Moreover, one has

$$\lim_{R \rightarrow +\infty} \left(\frac{c_p c_q}{c_r} \right)^{\frac{n}{2} \frac{1}{R-1}} = \frac{1}{(1-\lambda)^{n(1-\lambda)}} \frac{1}{\lambda^{n\lambda}}.$$

Hence, the right-hand side of inequality (1.11) tends to $\|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}$ when R tends to $+\infty$, and the left-hand side tends to $\|h\|_{L^1(\mathbb{R}^n)}$.

A similar computation will be done in the proof of Theorem 1.6.1.

For $p, q, r \geq 1$, Theorem 1.4.3 is a refinement of the classical *Young inequality*.

Theorem 1.4.4 (Young [85]). *Let $1 \leq p, q, r \leq +\infty$ such that $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ and $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$. Then,*

$$\frac{\|f \star g\|_{L^r}}{\|f\|_{L^p} \|g\|_{L^q}} \leq 1. \quad (1.12)$$

Another generalization of the Prékopa-Leindler inequality (1.8) is given by Barthe.

Theorem 1.4.5 (Barthe [5]). *Let $m \in \mathbb{N}^*$. For $i \in \{1, \dots, m\}$, let $n_i \in \mathbb{N}^*$ and let $\lambda_i \in [0, 1]$ such that $\sum_{i=1}^m \lambda_i n_i = n$, let $B_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n_i}$ be linear surjections, let $f_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_+$ be non-negative measurable functions. Then,*

$$\int_{\mathbb{R}^n} \sup_{x = \sum_{i=1}^m \lambda_i B_i^*(x_i)} f_1(x_1) \cdots f_m(x_m) dx \geq \sqrt{D} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f_i(x_i)^{\frac{1}{\lambda_i}} dx_i \right)^{\lambda_i}, \quad (1.13)$$

where the constant D is optimal and satisfies

$$D = \inf_{A_i \in \mathcal{S}^+} \frac{\det(\sum_{i=1}^m c_i B_i^* A_i B_i)}{\prod_{i=1}^m \det(A_i)^{c_i}}.$$

One derives the Prékopa-Leindler inequality (1.8) from the Barthe inequality (1.13) by taking for every $i \in \{1, \dots, m\}$, $n_i = n$ and $B_i = I_n$ the n -dimensional identity matrix; in this case, one has $D = 1$.

The most difficult part in the proof of inequality (1.8) is to prove it in dimension 1 since the general statement immediately follows by induction.

The Prékopa-Leindler inequality (1.8) can be seen as a reverse form of the classical Hölder inequality, which was in fact proved earlier by Rogers.

Theorem 1.4.6 (Rogers [73], Hölder [50]). *Let $\lambda \in [0, 1]$ and let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be non-negative measurable functions, then*

$$\int_{\mathbb{R}^n} f(x)g(x) dx \leq \left(\int_{\mathbb{R}^n} f(x)^{\frac{1}{1-\lambda}} dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)^{\frac{1}{\lambda}} dx \right)^{\lambda}. \quad (1.14)$$

A useful inequality on means follows from the Hölder inequality (1.14).

Lemma 1.4.7. *Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be measurable functions and $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ such that $\beta + \gamma \geq 0$ and $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$. Then, for every Borel measure μ ,*

$$\|fg\|_{L^\alpha(\mu)} \leq \|f\|_{L^\beta(\mu)} \|g\|_{L^\gamma(\mu)}.$$

Proof. Consider $\alpha > 0$. Since $\frac{\beta+\gamma}{\beta\gamma} = \frac{1}{\alpha}$ and $\beta + \gamma \geq 0$, then $\beta\gamma > 0$ and then $\beta > 0$ and $\gamma > 0$. We apply Hölder's inequality (1.14) to $\tilde{f} = f^\alpha$, $\tilde{g} = g^\alpha$, $p = \frac{\beta}{\alpha}$, $q = \frac{\gamma}{\alpha}$. It follows that

$$\int f^\alpha g^\alpha = \int \tilde{f} \tilde{g} \leq \left(\int \tilde{f}^p \right)^{\frac{1}{p}} \left(\int \tilde{g}^q \right)^{\frac{1}{q}} = \left(\int f^\beta \right)^{\frac{\alpha}{\beta}} \left(\int g^\gamma \right)^{\frac{\alpha}{\gamma}}.$$

The proof is similar for $\alpha < 0$. □

Corollary 1.4.8. *Let $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ such that $\beta + \gamma \geq 0$ and $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$. Then, for every $a, b, c, d > 0$ and $\lambda \in [0, 1]$,*

$$M_\lambda^\alpha(ac, bd) \leq M_\lambda^\beta(a, b) M_\lambda^\gamma(c, d). \quad (1.15)$$

Proof. Let $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ such that $\beta + \gamma \geq 0$ and $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$. Apply Lemma 1.4.7 to $\mu = (1-\lambda)\delta_{x_0} + \lambda\delta_{y_0}$ and f, g such that $f(x_0) = a$, $f(y_0) = b$, $g(x_0) = c$, $g(y_0) = d$. □

The Prékopa-Leindler inequality (1.8) is a functional version of the Brunn-Minkowski inequality (1.6) since one derives it by taking $f = 1_A$, $g = 1_B$ and $h = 1_{(1-\lambda)A + \lambda B}$ in

inequality (1.8). Indeed, it then follows that

$$|(1 - \lambda)A + \lambda B| \geq |A|^{1-\lambda}|B|^\lambda. \quad (1.16)$$

By replacing A with $\frac{A}{|A|^{\frac{1}{n}}}$, B with $\frac{B}{|B|^{\frac{1}{n}}}$ and λ with $\frac{|B|^{\frac{1}{n}}}{|A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}}$ in inequality (1.16), one obtains the Brunn-Minkowski inequality (1.6). More generally, with the same argument, one can show that an α -homogeneous $-\infty$ -concave function is in fact $\frac{1}{\alpha}$ -concave.

Let us see the power of the Brunn-Minkowski inequality (1.7) by showing that in fact this inequality implies the Borell-Brascamp-Lieb inequality (Theorem 1.4.1) and thus the Prékopa-Leindler inequality (1.8). Let us first start by proving this inequality in dimension 1.

Lemma 1.4.9 (Henstock, Macbeath [48]). *Let $\lambda \in [0, 1]$ and $\gamma \in [-\infty, +\infty]$. Let $f, g, h : \mathbb{R} \rightarrow \mathbb{R}_+$ be non-negative measurable functions satisfying for every $x, y \in \mathbb{R}^n$ such that $f(x)g(y) > 0$,*

$$h((1 - \lambda)x + \lambda y) \geq M_\lambda^\gamma(f(x), g(y)).$$

Then,

$$\int_{\mathbb{R}} h(x) dx \geq M_\lambda^\gamma(\|f\|_\infty, \|g\|_\infty) M_\lambda^1 \left(\frac{\int_{\mathbb{R}} f(x) dx}{\|f\|_\infty}, \frac{\int_{\mathbb{R}} g(x) dx}{\|g\|_\infty} \right).$$

Proof. Let us first assume that $\max(f) = \max(g)$. Notice that

$$\int_{\mathbb{R}} h(x) dx = \int_{\mathbb{R}} \left(\int_0^{h(x)} dt \right) dx = \int_0^{+\infty} |\{h \geq t\}| dt.$$

Since $\max(f) = \max(g)$, then for every $0 \leq t \leq \max(f)$, the sets $\{f \geq t\}$, $\{g \geq t\}$ are non-empty and for $t > \max(f)$ the sets $\{f \geq t\}$, $\{g \geq t\}$ are empty. Notice that for every $0 \leq t \leq \max(f)$

$$\{h \geq t\} \supset (1 - \lambda)\{f \geq t\} + \lambda\{g \geq t\}.$$

It follows from the Brunn-Minkowski inequality (1.7) in dimension 1, that for every $t \geq 0$

$$|\{h \geq t\}| \geq (1 - \lambda)|\{f \geq t\}| + \lambda|\{g \geq t\}|.$$

Finally,

$$\begin{aligned} \int_{\mathbb{R}} h(x) \, dx &= \int_{\mathbb{R}} |\{h \geq t\}| \, dt \geq (1 - \lambda) \int_{\mathbb{R}} |\{f \geq t\}| \, dt + \lambda \int_{\mathbb{R}} |\{g \geq t\}| \, dt \\ &= (1 - \lambda) \int_{\mathbb{R}} f(x) \, dx + \lambda \int_{\mathbb{R}} g(x) \, dx \\ &= M_{\lambda}^1 \left(\int_{\mathbb{R}} f(x) \, dx, \int_{\mathbb{R}} g(x) \, dx \right). \end{aligned}$$

For the general case, let us consider

$$\tilde{f}(x) = \frac{f(x)}{\|f\|_{\infty}}, \quad \tilde{g}(x) = \frac{g(x)}{\|g\|_{\infty}} \quad \text{and} \quad \tilde{h}(x) = \frac{h(x)}{M_{\lambda}^{\gamma}(\|f\|_{\infty}, \|g\|_{\infty})}.$$

Since

$$M_{\lambda}^{\gamma}(\tilde{f}(x)\|f\|_{\infty}, \tilde{g}(y)\|g\|_{\infty}) \geq M_{\lambda}^{\gamma}(\|f\|_{\infty}, \|g\|_{\infty}) \min(\tilde{f}(x), \tilde{g}(y)),$$

it follows that

$$h((1 - \lambda)x + \lambda y) \geq M_{\lambda}^{\gamma}(\|f\|_{\infty}, \|g\|_{\infty}) \min(\tilde{f}(x), \tilde{g}(y)).$$

Hence,

$$\tilde{h}((1 - \lambda)x + \lambda y) \geq \min(\tilde{f}(x), \tilde{g}(y)).$$

Since $\max(\tilde{f}) = \max(\tilde{g})$, it follows that

$$\int_{\mathbb{R}} h(x) \, dx \geq M_{\lambda}^{\gamma}(\|f\|_{\infty}, \|g\|_{\infty}) M_{\lambda}^1 \left(\frac{\int_{\mathbb{R}} f(x) \, dx}{\|f\|_{\infty}}, \frac{\int_{\mathbb{R}} g(x) \, dx}{\|g\|_{\infty}} \right).$$

□

Proof of BM inequality (1.7) \implies BBL inequality (Theorem 1.4.1). Let $\gamma \geq -\frac{1}{n}$ and let $\lambda \in [0, 1]$. For simplicity, we consider only three functions $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}_+$ satisfying for every $x, y \in \mathbb{R}^n$ such that $f(x)g(y) > 0$,

$$h((1 - \lambda)x + \lambda y) \geq M_{\lambda}^{\gamma}(f(x), g(y)).$$

In dimension 1 ($n = 1$), it is a direct consequence of Lemma (1.4.9) and inequal-

ity (1.15) since

$$\begin{aligned} \int_{\mathbb{R}} h(x) \, dx &\geq M_{\lambda}^{\gamma}(\|f\|_{\infty}, \|g\|_{\infty}) M_{\lambda}^1 \left(\frac{\int_{\mathbb{R}} f(x) \, dx}{\|f\|_{\infty}}, \frac{\int_{\mathbb{R}} g(x) \, dx}{\|g\|_{\infty}} \right) \\ &\geq M_{\lambda}^{\frac{\gamma}{1+\gamma}} \left(\int_{\mathbb{R}} f(x) \, dx, \int_{\mathbb{R}} g(x) \, dx \right), \end{aligned}$$

the last inequality being valid because $\gamma \geq -1$ by assumption.

In dimension n , with the change of variable $t = s^{\frac{1}{\gamma}}$, one has

$$\int_{\mathbb{R}^n} h(x) \, dx = \int_0^{+\infty} |\{h \geq t\}| \, dt = \frac{1}{|\gamma|} \int_0^{+\infty} |\{h \geq s^{\frac{1}{\gamma}}\}| s^{\frac{1}{\gamma}-1} \, ds.$$

Let us denote $F_s = \{f \geq s^{\frac{1}{\gamma}}\}$, $G_t = \{g \geq t^{\frac{1}{\gamma}}\}$ and $H_u = \{h \geq u^{\frac{1}{\gamma}}\}$. In the following, s and t are such that $s^{\frac{1}{\gamma}} \leq \|f\|_{\infty}$ and $t^{\frac{1}{\gamma}} \leq \|g\|_{\infty}$. If $x \in F_s$ and $y \in G_t$ then

$$h((1-\lambda)x + \lambda y) \geq M_{\lambda}^{\gamma}(f(x), g(y)) \geq ((1-\lambda)s + \lambda t)^{\frac{1}{\gamma}}.$$

It follows that $(1-\lambda)F_s + \lambda G_t \subset H_{(1-\lambda)s + \lambda t}$. From the Brunn-Minkowski inequality (1.7), one obtains

$$|H_{(1-\lambda)s + \lambda t}| \geq |(1-\lambda)F_s + \lambda G_t| \geq \left((1-\lambda)|F_s|^{\frac{1}{n}} + \lambda|G_t|^{\frac{1}{n}} \right)^n.$$

Hence,

$$\begin{aligned} |H_{(1-\lambda)s + \lambda t}| ((1-\lambda)s + \lambda t)^{\frac{1-\gamma}{\gamma}} &\geq M_{\lambda}^{\frac{1}{n}}(|F_s|, |G_t|) M_{\lambda}^{\frac{\gamma}{1-\gamma}}(s^{\frac{1-\gamma}{\gamma}}, t^{\frac{1-\gamma}{\gamma}}) \\ &\geq M_{\lambda}^{\frac{\gamma}{1+(n-1)\gamma}}(|F_s| s^{\frac{1-\gamma}{\gamma}}, |G_t| t^{\frac{1-\gamma}{\gamma}}), \end{aligned}$$

the last inequality being valid if $\gamma \in [-\frac{1}{n-1}, 1]$ according to inequality (1.15). Let us denote $F(s) = |F_s| s^{\frac{1-\gamma}{\gamma}}$, $G(t) = |G_t| t^{\frac{1-\gamma}{\gamma}}$ and $H(u) = |H_u| u^{\frac{1-\gamma}{\gamma}}$. We have just shown that

$$H((1-\lambda)s + \lambda t) \geq M_{\lambda}^{\frac{\gamma}{1+(n-1)\gamma}}(F(s), G(t)).$$

Since $\gamma \geq -\frac{1}{n}$, we can apply the theorem in dimension 1 to the functions F, G, H to obtain

$$\int_{\mathbb{R}} H(x) \, dx \geq M_{\lambda}^{\frac{\gamma}{1+n\gamma}} \left(\int_{\mathbb{R}} F(x) \, dx, \int_{\mathbb{R}} G(x) \, dx \right),$$

that is

$$\int_{\mathbb{R}^n} h(x) \, dx \geq M_\lambda^{\frac{\gamma}{1+n\gamma}} \left(\int_{\mathbb{R}^n} f(x) \, dx, \int_{\mathbb{R}^n} g(x) \, dx \right).$$

For the moment, we have seen that one can deduce the Borell-Brascamp-Lieb inequality (Theorem 1.4.1) from the Brunn-Minkowski inequality (1.7) in the range $\gamma \in [-\frac{1}{n}, 1]$.

To conclude the proof, one has to use a kind of homogeneity.

In the following, let us consider $\gamma \geq 1$. Since $M_\lambda^\gamma \geq M_\lambda^{-\frac{1}{n}}$, then

$$\int_{\mathbb{R}^n} h(x) \, dx \geq \min \left(\int_{\mathbb{R}^n} f(x) \, dx, \int_{\mathbb{R}^n} g(x) \, dx \right).$$

First, let us assume that $\int_{\mathbb{R}^n} f(x) \, dx = \int_{\mathbb{R}^n} g(x) \, dx = 1$. It follows that

$$\int_{\mathbb{R}^n} h(x) \, dx \geq \min(1, 1) = 1 = M_\lambda^{\frac{\gamma}{1+n\gamma}}(1, 1) = M_\lambda^{\frac{\gamma}{1+n\gamma}} \left(\int_{\mathbb{R}^n} f(x) \, dx, \int_{\mathbb{R}^n} g(x) \, dx \right).$$

For the general case, let us consider

$$F(x) = \frac{f \left(\left(\int f \right)^{\frac{\gamma}{1+n\gamma}} x \right)}{\left(\int f \right)^{\frac{1}{1+n\gamma}}}, \quad G(y) = \frac{g \left(\left(\int g \right)^{\frac{\gamma}{1+n\gamma}} y \right)}{\left(\int g \right)^{\frac{1}{1+n\gamma}}}, \quad H(z) = \frac{h(Cz)}{C^{\frac{1}{\gamma}}},$$

where $C = (1 - \lambda) \left(\int f \right)^{\frac{\gamma}{1+n\gamma}} + \lambda \left(\int g \right)^{\frac{\gamma}{1+n\gamma}}$. We check that

$$\int_{\mathbb{R}^n} F(x) \, dx = \int_{\mathbb{R}^n} \frac{f \left(\left(\int f \right)^{\frac{\gamma}{1+n\gamma}} x \right)}{\left(\int f \right)^{\frac{1}{1+n\gamma}}} \, dx = \frac{1}{\left(\int f \right)^{\frac{1}{1+n\gamma}}} \frac{1}{\left(\int f \right)^{\frac{n\gamma}{1+n\gamma}}} \int f = 1.$$

Similarly, one has $\int G = 1$. Moreover, by letting $\mu = \frac{\lambda}{C} \left(\int g \right)^{\frac{\gamma}{1+n\gamma}}$, one has

$$\begin{aligned} H((1 - \mu)u + \mu v) &= \frac{1}{C^{\frac{1}{\gamma}}} h(C((1 - \mu)u + \mu v)) \\ &= \frac{1}{C^{\frac{1}{\gamma}}} h \left((1 - \lambda) \left(\int f \right)^{\frac{\gamma}{1+n\gamma}} u + \lambda \left(\int g \right)^{\frac{\gamma}{1+n\gamma}} v \right) \\ &\geq \frac{1}{C^{\frac{1}{\gamma}}} \left((1 - \lambda) f \left(\left(\int f \right)^{\frac{\gamma}{1+n\gamma}} u \right)^\gamma + \lambda g \left(\left(\int g \right)^{\frac{\gamma}{1+n\gamma}} v \right)^\gamma \right)^{\frac{1}{\gamma}}, \end{aligned}$$

the last inequality being valid by assumption. It follows that

$$\begin{aligned} H((1 - \mu)u + \mu v) &\geq \left(\frac{1 - \lambda}{C} \left(\int f \right)^{\frac{\gamma}{1+n\gamma}} F(u)^\gamma + \frac{\lambda}{C} \left(\int g \right)^{\frac{\gamma}{1+n\gamma}} G(v)^\gamma \right)^{\frac{1}{\gamma}} \\ &= ((1 - \mu)F(u)^\gamma + \mu G(v)^\gamma)^{\frac{1}{\gamma}}. \end{aligned}$$

Since $\int F = \int G = 1$, one obtains $\int H \geq 1$, which is the desired inequality, recalling that

$$H(z) = \frac{h(Cz)}{C^{\frac{1}{\gamma}}},$$

where $C = (1 - \lambda) \left(\int f \right)^{\frac{\gamma}{1+n\gamma}} + \lambda \left(\int g \right)^{\frac{\gamma}{1+n\gamma}}$. □

For more information on the Brunn-Minkowski inequality and its influences on several mathematical theories, see the survey by Gardner [40]. See also the book by Schneider [77], as a general reference in Convex Geometry.

1.5 The Information theory

The Information theory was founded mainly by the engineers Nyquist [69], Hartley [47] and Shannon [78]. With an initial aim of comparing the *efficiency* of different communication systems, they define a measure which quantifies the transmission of information. As the result, a generic communication system has been thought of and taken up by Shannon:

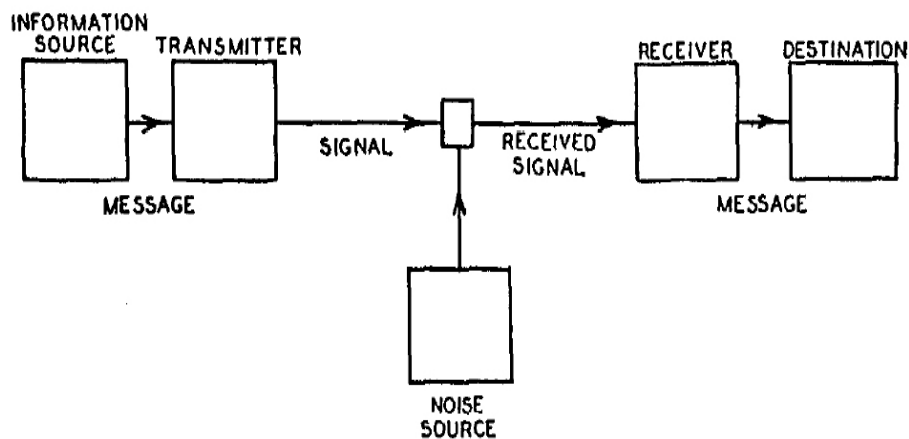


Figure 1.7: Generic communication system [78]

Shannon explained this schematic drawing in the following overview:

1. Information source: produces a message. For example, a sequence of letters as in a telegraph system.
2. Transmitter: operates on the message for transmission over a channel. For example, changing sounds into electrical currents as in the telephone.
3. Channel: medium used to transmit the message from transmitter to receiver. For example, wires, band of radio frequencies.
4. Receiver: performs the inverse operation of that done by the transmitter.
5. Destination: the person (or thing) for whom the message is intended.

A theory which excludes meaning

This theory of communication excludes meaning. It is restricted to the transfer of a message. The content of the message should have no influence on ways to transport it.

Shannon said in [78]:

These semantic aspects of communication are irrelevant to the engineering problem. The significant aspect is that the actual message is one selected from a set of possible messages. The system must be designed to operate for each possible selection, not just the one which will actually be chosen since this is unknown at the time of design.

Quantity of information

In fact, the idea of measuring the information has already been proposed by the engineer Ralph Hartley in [47]. Hartley introduced the concept of quantity of information. The idea is that for given events x_1, \dots, x_N with known probabilities p_1, \dots, p_N respectively, we then define the quantity of information associated to x_i by $-\log(p_i)$. This follows the idea that the higher the probability of an event, the less information this event brings. A natural remark is that one can consider arbitrary non-increasing function instead of $-\log$.

The logarithmic function: a natural choice?

Shannon in [78] reaffirmed what was earlier introduced by Hartley, why the logarithmic measure is more convenient:

1. Parameters of engineering importance such as time, number of relays, *etc.*, tend to vary linearly with the logarithm of the number of possibilities. For example, adding one relay to a group doubles the number of possible states of the relays.
2. It is nearer to our intuitive feeling as to the proper measure. This is closely related to 1. since we intuitively measure entities by linear comparison with common standards.
3. It is mathematically more suitable.

Shannon's entropy

Shannon introduced a new measure from the notion of quantity of information: the notion of *uncertainty*, what soon after became known as *entropy*. He was motivated to determine how much *choice* is involved in the selection of an event in a set of possible events whose only known information are probabilities of occurrence. According to

Shannon, such a measure, denoted by $H(p_1, \dots, p_N)$ with $p_1 + \dots + p_N = 1$, satisfies the following conditions:

1. H should be continuous in the p_i .
2. If all p_i are equal, $p_i = \frac{1}{N}$, then H should be a monotonic increasing function of N .
3. If a choice is broken down into two successive choices, the original H should be the weighted sum of the individual values of H .

Let us make a few remarks. The function H shall be denoted by $H^{(N)}$, where N is the number of variables of H . Also, the condition 3 can be written in the following way: For positive numbers n_1, \dots, n_r , and probabilities associated

$$p_1 = \frac{n_1}{N}, \dots, p_r = \frac{n_r}{N},$$

where $N = n_1 + \dots + n_r$, then

$$H^{(N)}\left(\frac{p_1}{n_1}, \dots, \frac{p_r}{n_r}\right) = H^{(r)}(p_1, \dots, p_r) + \sum_{i=1}^r p_i H^{(n_i)}\left(\frac{1}{n_i}, \dots, \frac{1}{n_i}\right).$$

where it is understood that

$$H^{(N)}\left(\frac{p_1}{n_1}, \dots, \frac{p_r}{n_r}\right) = H^{(N)}\left(\underbrace{\frac{p_1}{n_1}, \dots, \frac{p_1}{n_1}}_{n_1}, \dots, \underbrace{\frac{p_r}{n_r}, \dots, \frac{p_r}{n_r}}_{n_r}\right).$$

Theorem 1.5.1 (Shannon [78]). *The only functions H that satisfy the three conditions above are of the form:*

$$H(p_1, \dots, p_r) = -K \sum_{i=1}^r p_i \log(p_i),$$

where K is a positive constant.

Proof. Let $s, m \in \mathbb{N}^*$. We first show by induction on $m \in \mathbb{N}^*$ that

$$H^{(s^m)}\left(\frac{1}{s^m}, \dots, \frac{1}{s^m}\right) = mH^{(s)}\left(\frac{1}{s}, \dots, \frac{1}{s}\right). \quad (1.17)$$

For $m = 1$, it is direct. Let $m \geq 1$ and assume that

$$H^{(s^m)}\left(\frac{1}{s^m}, \dots, \frac{1}{s^m}\right) = mH^{(s)}\left(\frac{1}{s}, \dots, \frac{1}{s}\right).$$

Applying the condition 3 to $p_1 = \dots = p_r = \frac{1}{s}$ and $n_1 = \dots = n_r = s^m$, one obtains

$$\begin{aligned} H^{(s^{m+1})}\left(\frac{p_1}{n_1}, \dots, \frac{p_r}{n_r}\right) &= H^{(s^{m+1})}\left(\frac{1}{s^{m+1}}, \dots, \frac{1}{s^{m+1}}\right) \\ &= H^{(s)}\left(\frac{1}{s}, \dots, \frac{1}{s}\right) + H^{(s^m)}\left(\frac{1}{s^m}, \dots, \frac{1}{s^m}\right) \\ &= H^{(s)}\left(\frac{1}{s}, \dots, \frac{1}{s}\right) + mH^{(s)}\left(\frac{1}{s}, \dots, \frac{1}{s}\right) \\ &= (m+1)H^{(s)}\left(\frac{1}{s}, \dots, \frac{1}{s}\right). \end{aligned}$$

Let $t, s, n, m \in \mathbb{N}^*$ such that $t \leq s$ and

$$s^m \leq t^n \leq s^{m+1}.$$

Hence,

$$m \log(s) \leq n \log(t) \leq (m+1) \log(s).$$

Thus, after dividing by $n \log(s)$,

$$\left| \frac{\log(t)}{\log(s)} - \frac{m}{n} \right| \leq \frac{1}{n}. \quad (1.18)$$

For $N \geq 1$, we denote

$$A(N) = H^{(N)}\left(\frac{1}{N}, \dots, \frac{1}{N}\right).$$

By the condition 2, the function A is increasing and we then have

$$A(s^m) \leq A(t^n) \leq A(s^{m+1}).$$

From equality (1.17), we have

$$mA(s) \leq nA(t) \leq (m+1)A(s).$$

It follows from inequality (1.18) that

$$\left| \frac{\log(t)}{\log(s)} - \frac{A(t)}{A(s)} \right| \leq \frac{2}{n}.$$

We deduce that for every $t \in \mathbb{N}^*$,

$$A(t) = K \log(t),$$

with $K > 0$ by the condition 2.

Let p_1, \dots, p_r be rational numbers in $(0, 1]$. We write $p_i = \frac{n_i}{N}$ with $N = n_1 + \dots + n_r$. By the condition 3,

$$A(N) = H^{(r)}(p_1, \dots, p_r) + \sum_{i=1}^r p_i H^{(n_i)}\left(\frac{1}{n_i}, \dots, \frac{1}{n_i}\right).$$

Hence,

$$\begin{aligned} H^{(r)}(p_1, \dots, p_r) &= A(N) - \sum_{i=1}^r p_i A(n_i) \\ &= K \log(N) - K \sum_{i=1}^r p_i \log(n_i) \\ &= -K \sum_{i=1}^r p_i \log(p_i). \end{aligned}$$

Finally, by the condition 1, it follows that for every positive numbers p_1, \dots, p_r such that $p_1 + \dots + p_r = 1$,

$$H(p_1, \dots, p_r) = -K \sum_{i=1}^r p_i \log(p_i),$$

where K is a positive constant. □

Remark. The constant K can be omitted by considering a suitable base for the logarithm.

The theory

In the following, we present some basic facts about the Information theory, which are inspired by [29], [25] and [52].

Definition 1.5.2 (Discrete entropy). *Let X be a discrete random variable taking values x_1, \dots, x_n with probabilities p_1, \dots, p_n respectively. The entropy of X is*

$$H(X) = - \sum_{i=1}^n p_i \log(p_i),$$

with the convention that $0 \log(0) = 0$.

Example. Consider that X follows a Bernoulli distribution with parameter $p \in [0, 1]$. A direct computation shows that

$$H(X) = -p \log(p) - (1 - p) \log(1 - p).$$

1. If $p = 0$ or $p = 1$, then X is deterministic and as expected by historical approach of the entropy, one has $H(X) = 0$ (*no uncertainty*).
2. If $p = \frac{1}{2}$, then X is *as random as possible* and as expected, one has $H(X) = \log(2)$ which is maximal over all possible values of p (*the most uncertainty*) (see Figure (1.8) below).

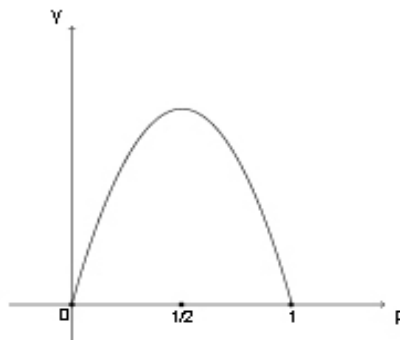


Figure 1.8: Graph of $H(X)$ with respect to p

Remark. As expected by historical approach, the discrete entropy of X does not depend on the image of X but on the probabilities of realizations.

Property 1.5.3. Let X be discrete random variable. Then, for every $a \neq 0$ and every $b \in \mathbb{R}$, one has

$$H(aX + b) = H(X).$$

Proof. The law of X and the law of $aX + b$ have the same probabilities of realization. \square

Definition 1.5.4 (Continuous entropy). Let X be a random vector in \mathbb{R}^n with probability density function f . The entropy of X is

$$H(X) = - \int f(x) \log(f(x)) dx.$$

Remark. The continuous entropy is also referred to as *differential entropy*. Since the entropy only depends on the probability density f , the entropy of X is sometimes written $H(f)$ instead of $H(X)$.

Property 1.5.5. Let X be a continuous random vector in \mathbb{R}^n . Then, for every $a \neq 0$ and every $b \in \mathbb{R}$, one has

$$H(aX + b) = H(X) + n \log(|a|).$$

Proof. Let us denote by f the density of X . Then the density of $aX + b$ is

$$g(u) = \frac{1}{|a|^n} f\left(\frac{u-b}{a}\right).$$

Hence,

$$\begin{aligned} H(aX + b) &= - \int_{\mathbb{R}^n} f\left(\frac{u-b}{a}\right) \log\left(\frac{1}{|a|^n} f\left(\frac{u-b}{a}\right)\right) \frac{1}{|a|^n} du \\ &= - \int_{\mathbb{R}^n} f(x) \log\left(\frac{1}{|a|^n} f(x)\right) dx \\ &= H(X) + n \log(|a|). \end{aligned}$$

□

Examples.

1. Let A be a compact subset of \mathbb{R}^n and X be a random vector uniformly distributed in A . Then,

$$H(X) = - \int \frac{1_A(x)}{|A|} \log\left(\frac{1_A(x)}{|A|}\right) dx = \frac{1}{|A|} \int_A \log(|A|) dx = \log(|A|).$$

2. Let X be a random variable following a Gaussian distribution with expectation m and variance σ^2 . Then, by translation invariance in Property 1.5.5, one can

consider that $m = 0$ and then

$$\begin{aligned} H(X) &= - \int \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \log \left(\frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \right) dx \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int \frac{x^2}{2\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx + \log(\sqrt{2\pi\sigma^2}) \\ &= \log(\sqrt{2\pi e}\sigma). \end{aligned}$$

3. Let X be a random vector following a Gaussian distribution with covariance matrix K , then

$$H(X) = \log(\sqrt{(2\pi e)^n \det(K)}).$$

Definition 1.5.6 (Relative entropy - Discrete case). *Let X and Y be two random variables with probabilities of realization p_1, \dots, p_n and q_1, \dots, q_n respectively. The relative entropy of X with respect to Y is*

$$D(X||Y) = \sum_{i=1}^n p_i \log \left(\frac{p_i}{q_i} \right).$$

Definition 1.5.7 (Relative entropy - Continuous case). *Let X and Y be two random vectors with probability densities f and g respectively. The relative entropy of X with respect to Y is*

$$D(X||Y) = \int f \log \left(\frac{f}{g} \right).$$

Remark. The relative entropy is also referred to as *Kullback-Leibler distance*. However, it is not a distance since it is not symmetric.

Proposition 1.5.8. *For every random variables X and Y ,*

$$D(X||Y) \geq 0.$$

Proof. By Jensen's inequality,

$$-D(X||Y) = \int f \log \left(\frac{g}{f} \right) \leq \log \left(\int g \right) = \log(1) = 0.$$

□

Definition 1.5.9 (Entropy power). Let X be a continuous random vector in \mathbb{R}^n . The entropy power of X is

$$N(X) = \frac{1}{2\pi e} e^{\frac{2}{n}H(X)},$$

where $H(X)$ is the entropy of X .

Property 1.5.10. Let X be a continuous random vector in \mathbb{R}^n . Then, for every $a \neq 0$ and every $b \in \mathbb{R}$, one has

$$N(aX + b) = a^2 N(X).$$

Proof. Using Property 1.5.5, one has

$$N(aX + b) = \frac{1}{2\pi e} e^{\frac{2}{n}H(aX+b)} = \frac{1}{2\pi e} e^{\frac{2}{n}(n \log(|a|) + H(X))} = a^2 N(X).$$

□

Definition 1.5.11 (Score function). Let X be a continuous random vector in \mathbb{R}^n with differentiable probability density f . The score function of X is

$$\rho_X(x) = \frac{\nabla f(x)}{f(x)}, \quad x \in \mathbb{R}^n.$$

Definition 1.5.12 (Fisher information). Let X be a continuous random vector in \mathbb{R}^n with score function ρ_X . The Fisher information of X is

$$I(X) = \mathbb{E}_X[|\rho_X|^2] = \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{f(x)} dx.$$

Property 1.5.13. Let X be a continuous random vector in \mathbb{R}^n . Then, for every $a \neq 0$ and every $b \in \mathbb{R}$, one has

$$I(aX + b) = \frac{1}{a^2} I(X).$$

Proof. Let us denote by f the density of X . Then the density of $aX + b$ is

$$g(u) = \frac{1}{|a|^n} f\left(\frac{u-b}{a}\right).$$

It follows that,

$$I(aX + b) = \int_{\mathbb{R}^n} \frac{\frac{1}{|a|^{2n+2}} |\nabla f\left(\frac{u-b}{a}\right)|^2}{\frac{1}{|a|^n} f\left(\frac{u-b}{a}\right)} du = \frac{1}{a^2} \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{f(x)} dx = \frac{1}{a^2} I(X).$$

□

Definition 1.5.14 (*p*-th Rényi entropy). Let $p \in (0, 1) \cup (1, +\infty)$. Let X be a continuous random vector in \mathbb{R}^n . The *p*-th Rényi entropy of X is

$$H_p(X) = \frac{1}{1-p} \log \left(\int_{\mathbb{R}^n} f^p(x) \, dx \right)$$

Property 1.5.15. Let X be a continuous random vector in \mathbb{R}^n . Then, for every $a \neq 0$ and every $b \in \mathbb{R}$, one has

$$H_p(aX + b) = H_p(X) + n \log(|a|).$$

Proof. The proof is similar to the proof of Property 1.5.5. □

Proposition 1.5.16. Let X be a continuous random vector in \mathbb{R}^n with probability density f . Let us denote by $\text{supp}(f)$ the support of f .

1. If $|\text{supp}(f)| < +\infty$, then $\lim_{p \rightarrow 0} H_p(X) = \log(|\text{supp}(f)|)$.
2. $\lim_{p \rightarrow 1} H_p(X) = H(X)$.
3. $\lim_{p \rightarrow +\infty} H_p(X) = -\log(\|f\|_\infty)$.

Proof. 1. One has,

$$\begin{aligned} \log \left(\int f^p(x) \, dx \right) &= \log \left(\int_{\{f>0\}} e^{p \log(f(x))} \, dx \right) \\ &= \log \left(\int_{\{f>0\}} [1 + p \log(f(x)) + o(p)] \, dx \right) \\ &= \log(|\text{supp}(f)|) + \log \left(1 + \frac{p}{|\text{supp}(f)|} \int [\log(f(x)) + o(1)] \, dx \right) \\ &= \log(|\text{supp}(f)|) + \frac{p}{|\text{supp}(f)|} \int [\log(f(x)) + o(1)] \, dx + o(p). \end{aligned}$$

Then,

$$\lim_{p \rightarrow 0} H_p(X) = \log(|\text{supp}(f)|).$$

2. Setting $P = p - 1$, one has

$$\begin{aligned}
 H_p(X) &= -\frac{1}{P} \log \left(\int f^P(x) f(x) \, dx \right) \\
 &= -\frac{1}{P} \log \left(\int [1 + P \log(f(x)) + o(p)] f(x) \, dx \right) \\
 &= -\frac{1}{P} \log \left(1 + P \int [\log(f(x)) + o(1)] f(x) \, dx \right) \\
 &= -\int [\log(f(x)) + o(1)] f(x) \, dx + o(1).
 \end{aligned}$$

Then,

$$\lim_{p \rightarrow 1} H_p(X) = H(X).$$

3. One has,

$$H_p(X) = \frac{1}{1-p} \log \left(\int_{\mathbb{R}^n} f^p(x) \, dx \right) = -\log \left(\int_{\mathbb{R}^n} f^{p-1}(x) f(x) \, dx \right)^{\frac{1}{p-1}}.$$

It follows that

$$\lim_{p \rightarrow +\infty} H_p(X) = -\log(\|f\|_\infty).$$

□

Distributions that maximize entropy

The next proposition reinforces Shannon's intuition about his notion of uncertainty and therefore the mathematical definition of Shannon's entropy is satisfactory in this sense.

Proposition 1.5.17. *Let A be a compact subset of \mathbb{R}^n . Among all random vectors compactly supported in A , the entropy is maximum for the uniform distribution on A .*

Proof. Let X be a random vector in \mathbb{R}^n supported in A and Y be a random vector following the uniform distribution on A . Then, using Proposition 1.5.8,

$$0 \leq D(X||Y) = \int_A f(x) \log(f(x)|A|) \, dx = -H(X) + \log(|A|) = -H(X) + H(Y).$$

□

The next proposition puts forward the Gaussian distribution in Information theory.

Proposition 1.5.18. *For a given variance, the Gaussian distribution maximizes the entropy.*

Proof. Let X be a random vector in \mathbb{R}^n with probability density f and covariance matrix (K_{ij}) and Y be a random variable following the Gaussian distribution ϕ with covariance matrix (K_{ij}) . Recall that

$$\phi(x) = \frac{1}{(2\pi)^{\frac{n}{2}} \det(K)} e^{-\frac{x^t K^{-1} x}{2}}.$$

By assumption,

$$\int f(x) x_i x_j \, dx = K_{ij} = \int \phi(x) x_i x_j \, dx.$$

Using Proposition 1.5.8, it follows that

$$0 \leq D(X||Y) = \int f \log(f) - \int f \log(\phi) = \int f \log(f) - \int \phi \log(\phi) = -H(X) + H(Y).$$

□

de Bruijn's identity

The de Bruijn identity links two important quantities in Information theory, the entropy and the Fisher information.

Proposition 1.5.19 (de Bruijn's identity [80]). *Let X be a continuous random vector in \mathbb{R}^n with finite covariances and Z be a standard Gaussian random vector independent of X . Then, for every $t \geq 0$,*

$$\frac{d}{dt} H(X + \sqrt{t}Z) = \frac{1}{2} I(X + \sqrt{t}Z). \quad (1.19)$$

Proof. Let us denote by f_t the density of $X + \sqrt{t}Z$. A direct calculation shows that f_t satisfies the heat equation:

$$\frac{\partial}{\partial t} f_t(x) = \frac{1}{2} \Delta f_t(x),$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ denotes the Laplace operator. It follows that

$$\begin{aligned}
\frac{d}{dt}H(X + \sqrt{t}Z) &= - \int \frac{\partial}{\partial t} f_t(x) \log(f_t(x)) dx - \int f_t(x) \frac{\partial}{\partial t} \log(f_t(x)) dx \\
&= - \int \frac{1}{2} \Delta f_t(x) \log(f_t(x)) dx \\
&= \frac{1}{2} \int \nabla f_t(x) \cdot \nabla \log(f_t(x)) dx \\
&= \frac{1}{2} \int \frac{\nabla f_t(x) \cdot \nabla f_t(x)}{f_t(x)} dx \\
&= \frac{1}{2} I(X + \sqrt{t}Z).
\end{aligned}$$

□

The Blachman-Stam inequality and the EPI

In this subsection, we present an interesting inequality – the Blachman-Stam inequality – that simply implies important inequalities in Information theory, such as the entropy power inequality (EPI), via de Bruijn’s identity (1.19). The entropy power inequality was established by Shannon in [78].

Theorem 1.5.20 (Entropy power inequality [78]). *Let X and Y be two independent continuous random vectors in \mathbb{R}^n . Then,*

$$N(X + Y) \geq N(X) + N(Y). \quad (1.20)$$

Notice that inequality (1.20) is equivalent to

$$H(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \geq \lambda H(X) + (1-\lambda)H(Y), \quad (1.21)$$

for every $\lambda \in [0, 1]$. Indeed, by replacing X with $\sqrt{\lambda}X$ and Y with $\sqrt{1-\lambda}Y$, $\lambda \in [0, 1]$, in inequality (1.20) and using concavity of the logarithm, one obtains inequality (1.21). Conversely, by replacing X with $\frac{X}{\sqrt{\lambda}}$ and Y with $\frac{Y}{\sqrt{1-\lambda}}$, where $\lambda = \frac{N(X)}{N(X)+N(Y)}$, in inequality (1.21), one obtains inequality (1.20).

We present a proof of the entropy power inequality (1.20) given by Blachman in [7]. The proof uses the Blachman-Stam inequality.

Theorem 1.5.21 (Blachman [7], Stam [80]). *Let X and Y be two independent continuous random vectors in \mathbb{R}^n . Then,*

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}. \quad (1.22)$$

For the proof of the Blachman-Stam inequality (1.22), we will follow the book of Johnson [52].

Lemma 1.5.22. *Let X and Y be two independent random vectors. Let us denote $Z = X + Y$. Let us denote ρ_X (resp. ρ_Y, ρ_Z) the score function of X (resp. Y, Z). Then,*

$$\rho_Z(z) = \mathbb{E}[\rho_X(X)|Z = z] = \mathbb{E}[\rho_Y(Y)|Z = z].$$

Proof. Let us denote by f (resp. g, h) the density of X (resp. Y, Z). One has

$$h(z) = \int f(x)g(z-x) dx.$$

Thus,

$$\begin{aligned} \frac{\partial h}{\partial z_i}(z) &= \int f(x) \frac{\partial}{\partial z_i} g(z-x) dx \\ &= - \int f(x) \frac{\partial}{\partial x_i} g(z-x) dx \\ &= \int \left(\frac{\partial}{\partial x_i} f(x) \right) g(z-x) dx. \end{aligned}$$

where one uses an integration by parts in the last equality. Hence,

$$\frac{\frac{\partial h}{\partial z_i}(z)}{h(z)} = \int \left(\frac{\partial}{\partial x_i} f(x) \right) \frac{g(z-x)}{h(z)} dx = \int \left(\frac{\frac{\partial}{\partial x_i} f(x)}{f(x)} \right) \frac{f(x)g(z-x)}{h(z)} dx.$$

Thus,

$$(\rho_Z(z))_i = \mathbb{E}[(\rho_X(X))_i | Z = z].$$

Using the same argument, one has

$$(\rho_Z(z))_i = \mathbb{E}[(\rho_Y(Y))_i | Z = z].$$

□

Lemma 1.5.23. *Let X and Y be two independent random vectors. Let $\lambda \in [0, 1]$. Then,*

$$I(X + Y) \leq \lambda^2 I(X) + (1 - \lambda)^2 I(Y).$$

Proof. From Lemma 1.5.22, one has

$$\rho_Z(z) = \mathbb{E}[\rho_X(X)|Z = z] = \mathbb{E}[\rho_Y(Y)|Z = z].$$

Thus, for every $\lambda \in [0, 1]$,

$$\rho_Z(z) = \mathbb{E}[\lambda \rho_X(X) + (1 - \lambda) \rho_Y(Y)|Z = z].$$

Hence,

$$\begin{aligned} I(X + Y) &= \mathbb{E}[\rho_Z(z)^2] \\ &= \mathbb{E}[\mathbb{E}[\lambda \rho_X(X) + (1 - \lambda) \rho_Y(Y)|Z = z]^2] \\ &\leq \mathbb{E}[\lambda^2 \rho_X(X)^2 + (1 - \lambda)^2 \rho_Y(Y)^2 + 2\lambda(1 - \lambda) \rho_X(X) \rho_Y(Y)] \\ &= \lambda^2 I(X) + (1 - \lambda)^2 I(Y). \end{aligned}$$

The last equality follows from

$$\mathbb{E}[\rho_X(X) \rho_Y(Y)] = \mathbb{E}[\rho_X(X)] \mathbb{E}[\rho_Y(Y)] \quad \text{and} \quad \mathbb{E}[\rho_X(X)] = 0.$$

□

Proof of Theorem 1.5.21. From Lemma 1.5.23, one has

$$I(X + Y) \leq \min_{\lambda \in [0, 1]} \lambda^2 I(X) + (1 - \lambda)^2 I(Y).$$

A direct study of the function $\lambda \mapsto \lambda^2 I(X) + (1 - \lambda)^2 I(Y)$ shows that the minimum is reached for $\lambda = I(Y)/(I(X) + I(Y))$. It follows that

$$\frac{1}{I(X + Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)}.$$

□

Two consequences follow from the Blachman-Stam inequality and de Bruijn's identity.

Proof of entropy power inequality (1.20). Let X_1, Y_1 be two independent random vectors and let $\lambda, t \in [0, 1]$. Let X_0, Y_0 be standard Gaussian random vectors independent of X_1, Y_1 and of each other.

Let us denote $X_t = \sqrt{t}X_1 + \sqrt{1-t}X_0$, $Y_t = \sqrt{t}Y_1 + \sqrt{1-t}Y_0$ and $V_t = \sqrt{\lambda}X_t + \sqrt{1-\lambda}Y_t$. One has

$$V_t = \sqrt{t}(\sqrt{\lambda}X_1 + \sqrt{1-\lambda}Y_1) + \sqrt{1-t}(\sqrt{\lambda}X_0 + \sqrt{1-\lambda}Y_0) = \sqrt{t}V_1 + \sqrt{1-t}V_0.$$

Let us denote $s(t) = H(V_t) - \lambda H(X_t) - (1-\lambda)H(Y_t)$. The entropy power inequality (1.20) is equivalent to $s(1) \geq 0$ (c.f. inequality (1.21)). Since V_0 is a standard Gaussian random vector, then $s(0) = 0$. It is then sufficient to prove that $s'(t) \geq 0$ to obtain the entropy power inequality. One has

$$s(t) = H(V_1 + \sqrt{\frac{1-t}{t}}V_0) - \lambda H(X_1 + \sqrt{\frac{1-t}{t}}X_0) - (1-\lambda)H(Y_1 + \sqrt{\frac{1-t}{t}}Y_0),$$

then

$$s'(t) = -\frac{1}{2t^2} \left(I(V_1 + \sqrt{\frac{1-t}{t}}V_0) - \lambda I(X_1 + \sqrt{\frac{1-t}{t}}X_0) - (1-\lambda)I(Y_1 + \sqrt{\frac{1-t}{t}}Y_0) \right).$$

It follows from Lemma (1.5.23) and the scaling property $I(X) = \alpha^2 I(\alpha X)$ that

$$2ts'(t) = -I(V_t) + \lambda I(X_t) + (1-\lambda)I(Y_t) \geq 0.$$

□

Corollary 1.5.24 (Concavity of entropy power). *Let X be a continuous random vector in \mathbb{R}^n and let Z be a standard Gaussian random vector independent of X . Then, the function $t \mapsto N(X + \sqrt{t}Z)$ is concave on \mathbb{R}_+ .*

Proof. Recall that $N(X + \sqrt{t}Z) = \frac{1}{2\pi e} e^{\frac{2}{n}H(X + \sqrt{t}Z)}$. Then,

$$\begin{aligned} \frac{d}{dt}N(X + \sqrt{t}Z) &= \frac{2}{n}N(X + \sqrt{t}Z)\frac{d}{dt}H(X + \sqrt{t}Z) \\ &= \frac{1}{n}N(X + \sqrt{t}Z)I(X + \sqrt{t}Z) \quad (\text{identity (1.19)}). \end{aligned}$$

$$\begin{aligned}\frac{d^2}{dt^2}N(X + \sqrt{t}Z) &= \frac{1}{n}N(X + \sqrt{t}Z) \left(\frac{d}{dt}I(X + \sqrt{t}Z) + \frac{1}{n}I(X + \sqrt{t}Z)^2 \right) \\ &\leq 0 \quad (\text{Proposition (1.6.3), proved thereafter}).\end{aligned}$$

□

Nowadays, the classical Information theory presented above is sometimes called *the Shannon Information theory* or *the Hartley-Shannon Information theory*.

1.6 Relationships between the Brunn-Minkowski theory and the Information theory

Many mathematicians have bridged the Brunn-Minkowski theory and the Information theory, notably with the precursor works of Costa, Cover, Dembo and Thomas in [24], [28], [29]. These links are realized through fundamental inequalities in each theory: the Brunn-Minkowski inequality (1.7) and the entropy power inequality (1.20). The following theorem unifies the Brunn-Minkowski inequality (1.7) and the entropy power inequality (1.20).

Theorem 1.6.1 (Rényi entropy inequality). *Let $\lambda \in [0, 1]$. Let $0 < r \leq +\infty$, $r \neq 1$. Let p, q such that $\frac{1}{p'} = \frac{\lambda}{r'}$ and $\frac{1}{q'} = \frac{1-\lambda}{r'}$, where p' is the conjugate of p defined by $\frac{1}{p} + \frac{1}{p'} = 1$, and similarly for q', r' . Let X, Y be two independent random vectors. Then,*

$$\begin{aligned} H_r(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) &= \lambda H_p(X) + (1-\lambda)H_q(Y) \\ &\geq H_r(\phi_{Id}) - \lambda H_p(\phi_{Id}) - (1-\lambda)H_q(\phi_{Id}), \end{aligned}$$

where ϕ_{Id} denotes the density of a Gaussian random vector whose covariance matrix is the Identity.

Proof. Let f be the density of $\sqrt{\lambda}X$ and g be the density of $\sqrt{1-\lambda}Y$. Notice that

$$\begin{aligned} H_p(\sqrt{\lambda}X) &= -p' \log(\|f\|_{L^p}) \\ H_q(\sqrt{1-\lambda}Y) &= -q' \log(\|g\|_{L^q}) \\ H_r(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) &= -r' \log(\|f \star g\|_{L^r}). \end{aligned}$$

The argument is similar for $r > 1$ or $0 < r < 1$. Let us assume that $r > 1$, then $r' \geq 0$. By applying Theorem 1.4.3, one has

$$\frac{\|f \star g\|_{L^r}}{\|f\|_{L^p} \|g\|_{L^q}} \leq \left(\frac{c_p c_q}{c_r} \right)^{\frac{n}{2}},$$

namely

$$-r' \log(\|f \star g\|_{L^r}) + r' \log(\|f\|_{L^p}) + r' \log(\|g\|_{L^q}) \geq -r' \frac{n}{2} \log \left(\frac{c_p c_q}{c_r} \right).$$

By assumption, $r' = p'\lambda = q'(1 - \lambda)$. Thus,

$$H_r(\sqrt{\lambda}X + \sqrt{1 - \lambda}Y) - \lambda H_p(\sqrt{\lambda}X) - (1 - \lambda)H_q(\sqrt{1 - \lambda}Y) \geq -r'\frac{n}{2} \log\left(\frac{c_p c_q}{c_r}\right),$$

Then, from homogeneity of the entropy,

$$\begin{aligned} & H_r(\sqrt{\lambda}X + \sqrt{1 - \lambda}Y) - \lambda H_p(X) - (1 - \lambda)H_q(Y) \\ & \geq -r'\frac{n}{2} \log\left(\frac{c_p c_q}{c_r}\right) + \frac{n}{2}\lambda \log(\lambda) + \frac{n}{2}(1 - \lambda) \log(1 - \lambda). \end{aligned}$$

Notice that

$$\begin{aligned} \log\left(\frac{c_p c_q}{c_r}\right) &= -\frac{\lambda}{1 - p} \log\left(\frac{1}{p}\right) - \frac{1 - \lambda}{1 - q} \log\left(\frac{1}{q}\right) + \frac{1}{1 - r} \log\left(\frac{1}{r}\right) \\ &\quad + \lambda \log(p') + (1 - \lambda) \log(q') - \log(r') \end{aligned}$$

and

$$H_p(\phi_{Id}) = \frac{n}{2} \left(\log(2\pi) + \frac{1}{1 - p} \log\left(\frac{1}{p}\right) \right).$$

Hence,

$$\begin{aligned} -r'\frac{n}{2} \log\left(\frac{c_p c_q}{c_r}\right) &= H_r(\phi_{Id}) - \lambda H_p(\phi_{Id}) - (1 - \lambda)H_q(\phi_{Id}) \\ &\quad + \frac{n}{2}(\lambda \log(p') + (1 - \lambda) \log(q') + \log(r')). \end{aligned}$$

using $r' = p'\lambda = q'(1 - \lambda)$, one has

$$\frac{n}{2}(\lambda \log(p') + (1 - \lambda) \log(q') + \log(r')) = -\frac{n}{2}(\lambda \log(\lambda) + (1 - \lambda) \log(1 - \lambda)).$$

It follows that

$$\begin{aligned} -r'\frac{n}{2} \log\left(\frac{c_p c_q}{c_r}\right) + \frac{n}{2}\lambda \log(\lambda) + \frac{n}{2}(1 - \lambda) \log(1 - \lambda) \\ = H_r(\phi_{Id}) - \lambda H_p(\phi_{Id}) - (1 - \lambda)H_q(\phi_{Id}). \end{aligned}$$

Finally,

$$\begin{aligned} & H_r(\sqrt{\lambda}X + \sqrt{1 - \lambda}Y) - \lambda H_p(X) - (1 - \lambda)H_q(Y) \\ & \geq H_r(\phi_{Id}) - \lambda H_p(\phi_{Id}) - (1 - \lambda)H_q(\phi_{Id}). \end{aligned}$$

□

Consequence 1. [Brunn-Minkowski inequality (1.7)]

Let $\lambda \in [0, 1]$. Let X and Y be two independent random vectors. Let us recall that

$$\frac{p-1}{p} = \lambda \frac{r-1}{r} \quad \text{and} \quad \frac{q-1}{q} = (1-\lambda) \frac{r-1}{r}.$$

It follows that

$$\left(\frac{\lambda}{1-p} + \frac{1-\lambda}{1-q} \right) \log \left(\frac{1}{r} \right) = \frac{1}{1-r} \log \left(\frac{1}{r} \right) + \frac{r}{1-r} \log \left(\frac{1}{r} \right).$$

Hence, one has

$$\begin{aligned} & \lim_{r \rightarrow 0} H_r(\phi_{Id}) - \lambda H_p(\phi_{Id}) - (1-\lambda) H_q(\phi_{Id}) \\ &= \frac{n}{2} \lim_{r \rightarrow 0} \left(\frac{1}{1-r} \log \left(\frac{1}{r} \right) - \frac{\lambda}{1-p} \log \left(\frac{1}{p} \right) - \frac{1-\lambda}{1-q} \log \left(\frac{1}{q} \right) \right) \\ &= \frac{n}{2} \lim_{r \rightarrow 0} \left(\frac{-r}{1-r} \log \left(\frac{1}{r} \right) - \frac{\lambda}{1-p} \log \left(\frac{r}{p} \right) - \frac{1-\lambda}{1-q} \log \left(\frac{r}{q} \right) \right) \\ &= \frac{n}{2} (-\lambda \log(\lambda) - (1-\lambda) \log(1-\lambda)). \end{aligned}$$

By replacing X with $\sqrt{\lambda}X$ and Y with $\sqrt{1-\lambda}Y$, one obtains

$$H_0(\lambda X + (1-\lambda)Y) \geq \lambda H_0(X) + (1-\lambda) H_0(Y).$$

By taking X uniformly distributed on $A \subset \mathbb{R}^n$ and Y uniformly distributed on $B \subset \mathbb{R}^n$, one deduces that

$$\log(|\lambda A + (1-\lambda)B|) \geq \lambda \log(|A|) + (1-\lambda) \log(|B|).$$

Consequence 2. [Entropy power inequality (1.20)]

Let $\lambda \in [0, 1]$. Let X and Y be two independent random vectors. Since $H(\phi_{Id})$ is finite, by letting r go to 1 in Theorem 1.6.1, one has

$$H(\sqrt{\lambda}X + \sqrt{1-\lambda}Y) \geq \lambda H(X) + (1-\lambda) H(Y).$$

A kind of dictionary

The objects of each theory are in correspondence with each other: the compact sets in the Brunn-Minkowski theory correspond to the random vectors in the Information theory, the Gaussian random vectors play the same role as the Euclidean balls, the entropy power N corresponds to the $1/n$ power of the volume $|\cdot|^{1/n}$ and, by taking logarithms, the entropy H is the analogue of the logarithm of the volume $\log(|\cdot|)$. It could be of interest to understand what can be the analogue of independence in the Brunn-Minkowski theory.

Let us summarize these analogies in the following table:

Information theory	Brunn-Minkowski theory
Random vector X	Compact A
Vector sum $X + Y$	Minkowski sum $A + B$
Gaussian Z	Euclidean ball B_2^n
$N(X)$	$ A ^{\frac{1}{n}}$
$H(X)$	$\log(A)$
Independence	Convexity?

Table 1.1: Analogous objects

Notice that if X has a uniform distribution on a measurable set $A \subset \mathbb{R}^n$, then $H(X) = \log(|A|)$.

Isoperimetric-type inequalities in Information theory

Since there are similarities between the Brunn-Minkowski inequality (1.6) and the entropy power inequality (1.20), it is natural to try to derive new inequalities in Information theory from the entropy power inequality, in the same way that we have derived the isoperimetric inequality from the Brunn-Minkowski inequality in Chapter 1, Section 1.3.

Proposition 1.6.2 (Isoperimetric inequality for entropies [29]). *Let X be a continuous*

random vector. Then,

$$\frac{1}{n}I(X)N(X) \geq 1.$$

Proof. By taking $Y = \sqrt{t}Z$ in inequality (1.20), where Z is a standard Gaussian random vector independent of X , and using $N(\sqrt{t}Z) = t$, one has

$$\frac{1}{t}(N(X + \sqrt{t}Z) - N(X)) \geq 1,$$

and by letting t go to 0, one obtains

$$\left. \frac{d}{dt}N(X + \sqrt{t}Z) \right|_{t=0} \geq 1.$$

Using de Bruijn's identity (1.19), it follows that

$$\frac{1}{n}I(X)N(X) \geq 1.$$

□

Proposition 1.6.3 (Isoperimetric inequality for Fisher information [29]). *Let X be a continuous random vector and Z be a standard Gaussian random vector independent of X . Then,*

$$\frac{1}{n}I(X)^2 + \left. \frac{d}{dt}I(X + \sqrt{t}Z) \right|_{t=0} \leq 0.$$

Proof. By taking $Y = \sqrt{t}Z$ in inequality (1.22), one has

$$\frac{1}{t} \left(\frac{1}{I(X + \sqrt{t}Z)} - \frac{1}{I(X)} \right) \geq \frac{1}{n},$$

and by letting t go to 0, one has

$$\left. \frac{d}{dt} \left(\frac{1}{I(X + \sqrt{t}Z)} \right) \right|_{t=0} \geq \frac{1}{n}.$$

In other words, one has

$$\frac{1}{n}I(X)^2 + \left. \frac{d}{dt}I(X + \sqrt{t}Z) \right|_{t=0} \leq 0.$$

□

1.7 Convex measures

The Brunn-Minkowski inequality (1.7) can be extended by considering measures other than the Lebesgue measure. Borell [14], [15] developed the theory of convex measures whose definition is based on the Brunn-Minkowski inequality (1.7). More precisely, Borell defined the notion of *s-concave measure* as a measure μ satisfying the following Brunn-Minkowski-type inequality:

$$\mu((1 - \lambda)A + \lambda B) \geq ((1 - \lambda)\mu(A)^s + \lambda\mu(B)^s)^{\frac{1}{s}}, \quad (1.23)$$

for every compact subsets A, B of \mathbb{R}^n such that $\mu(A)\mu(B) > 0$ and for every $\lambda \in [0, 1]$. The Lebesgue measure is then seen as a $\frac{1}{n}$ -concave measure. The parameter s belongs to $[-\infty; +\infty]$ and the limit cases are interpreted by continuity. More precisely, if $s = +\infty$ the right-hand side in (1.23) is $\max(\mu(A), \mu(B))$; if $s = 0$ the right-hand side is $\mu(A)^{1-\lambda}\mu(B)^\lambda$ and corresponds to log-concave measures; if $s = -\infty$ the right-hand side is $\min(\mu(A), \mu(B))$. From the inequality on means it follows that an s -concave measure is r -concave for every $r \leq s$. Thus, every s -concave measure is $-\infty$ -concave. The $-\infty$ -concave measures are also called *convex measures*.

Borell characterized s -concave measures in terms of a precise concavity property of its density.

Theorem 1.7.1 (Borell [15]). *Let μ be a measure in \mathbb{R}^n . Let H be the least affine subspace which contains the support of μ and set n_0 the dimension of H . Then, for $-\infty \leq s \leq \frac{1}{n_0}$, μ is s -concave if and only if μ admits a density f with respect to the Lebesgue measure on H , where f is γ -concave with $\gamma = s/(1 - sn_0) \in [-\frac{1}{n_0}, +\infty]$. Moreover, if $s > 1$, then μ is s -concave if and only if μ is a Dirac measure.*

Theorem 1.7.1 allows us to give more examples of s -concave measures. The case $s = 0$ corresponds to the log-concave measures. The most famous example of a log-concave measure is the standard Gaussian measure

$$d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} dx, \quad x \in \mathbb{R}^n$$

where $|\cdot|$ stands for the Euclidean norm. These measures are of interest. For example, the isoperimetric inequality has been established for the Gaussian measure $d\gamma_n$ by Borell in [16] and independently by Sudakov and Cirel'son in [82], which states that among

sets of a given Gauss measure, half-spaces minimize the Gauss surface area. Thereafter, Kannan, Lovász and Simonovits in [56] formulated their famous conjecture: for the uniform measure on a convex body (which can be extended to arbitrary log-concave measures) half-spaces are *nearly* minimizers. In dimension 1, Bobkov (see [8] and reference therein) proved that among sets of a given measure, half-lines minimize the surface area. More recently, a refined statement has been established by Cianchi *et al.* in [20]: in dimension n , a set of a given Gauss measure and *almost* minimal Gauss surface area is necessarily *close* to being a half-space. This result has been extended by De Castro in [27] for all log-concave probability measures in dimension 1.

1.8 Publications

My thesis is structured around the following three publications, which are available on my personal web page or on the open archives HAL and arXiv.

1. *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory*, (with Matthieu Fradelizi), *Advances in Applied Mathematics* 57 (2014), 1-20.
Described in Chapter 2.
2. *Concavity properties of extensions of the parallel volume*, *Mathematika* (2015), doi:10.1112/S0025579314000369.
Described in Chapter 3 and Chapter 4.
3. *On the improvement of concavity of convex measures*, *Proceedings of the American Mathematical Society* (2015), doi: <http://dx.doi.org/10.1090/proc/12694>.
Described in Chapter 4.

Chapter 2

The analogue of the concavity of entropy power in the Brunn-Minkowski theory

In this chapter, I present my paper *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory* [37], written in collaboration with Matthieu Fradelizi, and published in *Advances in Applied Mathematics*. In this paper, we investigate a conjecture of Costa and Cover [24], and we study its relationships to geometric inequalities such as the isoperimetric inequality and the Brunn-Minkowski inequality.

2.1 The conjecture of Costa-Cover

First, let us explain the origin of the conjecture of Costa-Cover. Costa and Cover noticed the similarity between the entropy power and the Brunn-Minkowski inequalities: for every independent random vectors X, Y in \mathbb{R}^n , with finite entropy and for every compact sets A and B in \mathbb{R}^n one has

$$N(X + Y) \geq N(X) + N(Y) \quad \text{and} \quad |A + B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}},$$

where $|\cdot|$ denotes the n -dimensional Lebesgue measure and

$$N(X) = \frac{1}{2\pi e} e^{\frac{2}{n}H(X)}$$

denotes the entropy power of X . Recall that for X with density f the entropy of X is $H(X) = -\int f \ln f$ if the integral exists and $H(X) = -\infty$ otherwise. Applying the Brunn-Minkowski inequality to $B = \varepsilon B_2^n$ and letting ε tend to 0 one obtain the classical isoperimetric inequality

$$\frac{|\partial A|}{|A|^{\frac{n-1}{n}}} \geq n|B_2^n|^{\frac{1}{n}} = \frac{|\partial B_2^n|}{|B_2^n|^{\frac{n-1}{n}}},$$

where the outer Minkowski surface area is defined by

$$|\partial A| = \lim_{\varepsilon \rightarrow 0^+} \frac{|A + \varepsilon B_2^n| - |A|}{\varepsilon},$$

whenever the limit exists. In the same way, Costa and Cover applied the entropy power inequality to $Y = \sqrt{\varepsilon}G$, where G is a standard Gaussian random vector (the $\sqrt{\varepsilon}$ comes from the homogeneity of entropy power $N(\sqrt{\varepsilon}X) = \varepsilon N(X)$). Then by letting ε tend to 0 and using de Bruijn's identity

$$\frac{d}{dt}H(X + \sqrt{t}G) = \frac{1}{2}I(X + \sqrt{t}G),$$

which states that the Fisher information (denoted by I) is the derivative of the entropy along the heat semi-group, they obtained the following *isoperimetric inequality for entropy*

$$N(X)I(X) \geq n.$$

Notice that this inequality is equivalent to the Log-Sobolev inequality for the Gaussian measure, see [2] chapter 9.

This analogy between the results of the Information theory and the Brunn-Minkowski theory was later extended and further explained and unified through Young's inequality by Dembo [28] and later on by Dembo, Cover and Thomas [29]. Then, Szarek and Voiculescu [83] deduced the entropy power inequality from a restricted Brunn-Minkowski inequality. Each of these theories deal with a fundamental inequality, the Brunn-Minkowski inequality for the Brunn-Minkowski theory and the entropy power inequality for the Information theory. The objects of each theories are in correspondence with each other: to the compact sets in the Brunn-Minkowski theory correspond the random vectors in the Information theory, the Gaussian random vectors play the same role as the Euclidean balls, the entropy power N corresponds to the $1/n$ power of the volume $|\cdot|^{1/n}$ and, taking logarithms, the entropy H is the analogue of the logarithm of the volume $\log |\cdot|$. Hence one can conjecture that properties of one theory fit into the

other theory.

Thus, Costa and Cover [24], as an analogue of the concavity of entropy power with added Gaussian noise, which states that

$$t \mapsto N(X + \sqrt{t}G)$$

is a concave function (see [23] and [84]), formulated the following conjecture.

Conjecture 2.1.1 (Costa-Cover [24]). *Let A be a bounded measurable set in \mathbb{R}^n then the function $t \mapsto |A + tB_2^n|^{\frac{1}{n}}$ is concave on \mathbb{R}_+ .*

Notice that Guleryuz, Lutwak, Yang and Zhang [45] also pursued these analogies between the two theories and more recently, Bobkov and Madiman [12] established an analogue in Information theory of the Milman's reverse Brunn-Minkowski inequality.

2.2 Basic properties of the parallel volume and links with geometric inequalities

Notations

We work in the Euclidean space \mathbb{R}^n , $n \geq 1$, equipped with the ℓ_2^n norm $|\cdot|$, whose closed unit ball is denoted by B_2^n and canonical basis is $\{e_1, \dots, e_n\}$. We also denote by $|\cdot|$ the Lebesgue measure in \mathbb{R}^n . For non-empty sets A, B in \mathbb{R}^n we define their *Minkowski sum*

$$A + B = \{a + b : a \in A, b \in B\}.$$

We denote by $\text{int}(A)$, \bar{A} , ∂A , $\text{conv}(A)$ respectively the interior, the closure, the boundary, the convex hull of the set A . A function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is $\frac{1}{n}$ -concave if $f^{\frac{1}{n}}$ is concave on its support.

A set B is a *convex body* if B is a compact convex set of \mathbb{R}^n with non-empty interior. If 0 is in the interior of B , then the gauge associated to B is the function $\|\cdot\|_B$ defined by $\|x\|_B = \inf\{t > 0 : x \in tB\}$, for every $x \in \mathbb{R}^n$. Let A be a bounded measurable subset of \mathbb{R}^n . For $x \in \mathbb{R}^n$, we set $d_B(x, A) = \inf\{\|x - y\|_B : y \in A\}$ and we simply denote $d(x, A) = d_{B_2^n}(x, A)$. We denote by $V_{A,B}$ the function defined for $t \geq 0$ by

$$V_{A,B}(t) = |A + tB|.$$

For $B = B_2^n$, we simply denote $V_A = V_{A, B_2^n}$ the (outer) parallel volume function of A defined on \mathbb{R}_+ by

$$V_A(t) = |A + tB_2^n|.$$

The outer Minkowski surface area $|\partial A|$ of A may be defined using V_A : if the function V_A admits a right derivative at 0 then one has

$$(V_A)'_+(0) = \lim_{t \rightarrow 0^+} \frac{|A + tB_2^n| - |A|}{t} = |\partial A|.$$

Connectedness properties of the parallel set

Let A be a bounded measurable subset of \mathbb{R}^n and B be a convex body in \mathbb{R}^n , then for every $t > 0$ the set $A + tB$ has a finite number of connected components and this number is non-increasing as a function of t .

Indeed, let $t > 0$ and C be a connected component of $A + tB$. Let $x \in C$, then there exists $a \in A$ such that $x \in a + tB$. Moreover $a + tB$ is connected, hence $a + tB \subset C$ since C is the connected component of x . Thus $|C| \geq |tB| > 0$. Since $|A + tB|$ is finite and equal to the volume of the disjoint union of its connected components, there is a finite number of them.

Let $0 < t_0 \leq t_1$. Denote by C_1, \dots, C_N the connected components of $A + t_0B$. One has $A + t_1B = \cup_{i=1}^N (C_i + (t_1 - t_0)B)$ and since $C_i + (t_1 - t_0)B$ is connected, it follows that the number of connected components of $A + t_1B$ is at most N .

Regularity properties of the parallel volume

Let A be a compact subset of \mathbb{R}^n and B be a convex body in \mathbb{R}^n containing 0 in its interior. The function $d_B(\cdot, A)$ is Lipschitz, hence from Federer's co-area formula [31], one has

$$V_{A,B}(t) = |A + tB| = |A| + \int_0^t \mathcal{H}^{n-1}(\{x : d_B(x, A) = s\}) ds, \quad (2.1)$$

where \mathcal{H}^{n-1} denotes the $(n - 1)$ -dimensional Hausdorff measure. Therefore the function $V_{A,B}$ is absolutely continuous on \mathbb{R}_+ .

Notice that for every bounded measurable subset A of \mathbb{R}^n and every $0 < s < t$, one has

$$\overline{A} + sB \subset A + tB \subset \overline{A} + tB.$$

From the continuity of $V_{\bar{A},B}$, one has that $|A + tB| = |\bar{A} + tB|$ for $t > 0$. Hence we may assume in the following that A is compact.

Stachó [79] proved a better regularity for $V_{A,B}$, he proved namely that the function $V_{A,B}$ is a n -Kneser function, which means that for every $0 < t_0 \leq t_1$ and every $\lambda \geq 1$, one has

$$V_{A,B}(\lambda t_1) - V_{A,B}(\lambda t_0) \leq \lambda^n (V_{A,B}(t_1) - V_{A,B}(t_0)). \quad (2.2)$$

Stachó deduced that for every $0 < t_0 < t_1$, the function

$$t \mapsto V_{A,B}(t) - t^n \frac{V_{A,B}(t_1) - V_{A,B}(t_0)}{t_1^n - t_0^n}$$

is concave on $[t_1, +\infty)$. Thus $V_{A,B}$ admits right and left derivatives at every $t > 0$, which satisfy

$$(V_{A,B})'_+(t) \leq (V_{A,B})'_-(t) \quad (2.3)$$

and these two derivatives coincide for all $t > 0$ outside a countable set. Hence the outer Minkowski surface area of $A + tB_2^n$ exists for every $t > 0$ and one has

$$|\partial(A + tB_2^n)| = \lim_{\varepsilon \rightarrow 0^+} \frac{|A + tB_2^n + \varepsilon B_2^n| - |A + tB_2^n|}{\varepsilon} = (V_A)'_+(t). \quad (2.4)$$

In Proposition 2.3.8 below, we show that the function V_A is continuously differentiable on $[\text{diam}(A), +\infty)$. If A is convex or with sufficiently regular boundary then the equality (2.4) also holds for $t = 0$. For precise statements and comparisons between the outer Minkowski surface area and other measurements of $\partial(A + tB_2^n)$, like the Hausdorff measure, see [1].

Proposition 2.2.1. *Let A and B be compact subsets of \mathbb{R}^n with B convex, then the function $(s, t) \mapsto |sA + tB|$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$. Moreover the functions*

$$t \mapsto |A + tB| - t^n |B| \quad \text{and} \quad s \mapsto |sA + B| - s^n |A|$$

are non-decreasing. In particular, the function $(s, t) \mapsto |sA + tB|$ is non-decreasing in each coordinate.

Proof. Let us prove the continuity. Let $0 \leq t \leq t'$. Let $r > 0$ be such that $A \subset rB_2^n$ and

$B \subset rB_2^n$. Then we have

$$|A + tB| \leq |A + t'B| \leq |A + tB + r(t' - t)B_2^n|.$$

From (2.1) the function $t' \mapsto |A + tB + t'rB_2^n|$ is continuous at 0, thus the function $t \mapsto |A + tB|$ is continuous on \mathbb{R}_+ . Since for $s > 0$ and $t \geq 0$

$$|sA + tB| = s^n \left| A + \frac{t}{s}B \right|$$

then $(s, t) \mapsto |sA + tB|$ is continuous on $\mathbb{R}_+^* \times \mathbb{R}_+$. We also have for every $s \geq 0$ and $t \geq 0$

$$|tB| \leq |sA + tB| \leq |srB_2^n + tB|$$

so $(s, t) \mapsto |sA + tB|$ is continuous on $\{0\} \times \mathbb{R}_+$. It follows that the function $(s, t) \mapsto |sA + tB|$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

The monotonicity follows from (2.2). Indeed, The inequality (2.2) may be written in a different way, as follows

$$|A + \lambda t_1 B| - |A + \lambda t_0 B| \leq |\lambda A + \lambda t_1 B| - |\lambda A + \lambda t_0 B|.$$

Changing variables, it also means that for every $0 < s_0 \leq s_1$ and $0 < t_0 \leq t_1$

$$|s_0 A + t_1 B| - |s_0 A + t_0 B| \leq |s_1 A + t_1 B| - |s_1 A + t_0 B|.$$

Applied first to $s_1 = 1$ and $s_0 \rightarrow 0$, and then to $t_1 = 1$ and $t_0 \rightarrow 0$, we deduce that the functions

$$t \mapsto V_{A,B}(t) - t^n |B| \quad \text{and} \quad s \mapsto |sA + B| - s^n |A|$$

are non-decreasing. In particular, the function $(s, t) \mapsto |sA + tB|$ is non-decreasing in each coordinate. □

Remark. If A and B are arbitrary compact sets, it is not necessarily true that the function $V_{A,B}$ is non-decreasing as can be seen from the example of $A = \{0; 4\}$ and $B = [-5, -3] \cup [3, 5]$.

Links with geometric inequalities

Let us connect the Costa-Cover conjecture with the Brunn-Minkowski inequality and the isoperimetric inequality. We first establish that the conjecture of Costa-Cover has many equivalent reformulations.

Proposition 2.2.2. *Let A and B be compact sets in \mathbb{R}^n , with B convex. The following properties are equivalent.*

- (i) $t \mapsto |A + tB|^{\frac{1}{n}}$ is concave on \mathbb{R}_+ .
- (ii) $s \mapsto |sA + B|^{\frac{1}{n}}$ is concave on \mathbb{R}_+ .
- (iii) $\lambda \mapsto |(1 - \lambda)A + \lambda B|^{\frac{1}{n}}$ is concave on $[0, 1]$.
- (iv) $(s, t) \mapsto |sA + tB|^{\frac{1}{n}}$ is concave on $\mathbb{R}_+ \times \mathbb{R}_+$.

Proof. (iv) \implies (i), (iv) \implies (ii) and (iv) \implies (iii) are clear. Let us prove that (i) \implies (iv), a similar argument easily shows that (ii) \implies (iv) and (iii) \implies (iv). Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $g : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be defined by $f(t) = |A + tB|^{\frac{1}{n}}$ and $g(s, t) = |sA + tB|^{\frac{1}{n}}$, for every $s, t \in \mathbb{R}_+$. For every $t \geq 0$ and $s > 0$, we have, from the homogeneity of the volume

$$g(s, t) = sf\left(\frac{t}{s}\right).$$

Thus for every $\lambda \in [0, 1]$, $s_1, s_2 \in (0, +\infty)$ and $t_1, t_2 \in \mathbb{R}_+$ we have

$$g((1 - \lambda)s_1 + \lambda s_2, (1 - \lambda)t_1 + \lambda t_2) = ((1 - \lambda)s_1 + \lambda s_2)f\left(\frac{(1 - \lambda)t_1 + \lambda t_2}{(1 - \lambda)s_1 + \lambda s_2}\right).$$

Using the concavity of f , we deduce that

$$\begin{aligned} f\left(\frac{(1 - \lambda)t_1 + \lambda t_2}{(1 - \lambda)s_1 + \lambda s_2}\right) &= f\left(\frac{(1 - \lambda)s_1 \frac{t_1}{s_1} + \lambda s_2 \frac{t_2}{s_2}}{(1 - \lambda)s_1 + \lambda s_2}\right) \\ &\geq \frac{(1 - \lambda)s_1 f\left(\frac{t_1}{s_1}\right) + \lambda s_2 f\left(\frac{t_2}{s_2}\right)}{(1 - \lambda)s_1 + \lambda s_2} \\ &= \frac{(1 - \lambda)g(s_1, t_1) + \lambda g(s_2, t_2)}{(1 - \lambda)s_1 + \lambda s_2}. \end{aligned}$$

We deduce that g is concave on $(\mathbb{R}_+^*)^2$. Moreover, g is continuous on $(\mathbb{R}_+)^2$ by Proposition 2.2.1. Hence g is concave on $(\mathbb{R}_+)^2$. \square

Remark. Notice that if for two fixed compact sets A and B , with B convex, the assertion (iii) of Proposition 2.2.2 holds true then for every $\lambda \in [0, 1]$,

$$|(1 - \lambda)A + \lambda B|^{\frac{1}{n}} \geq (1 - \lambda)|A|^{\frac{1}{n}} + \lambda|B|^{\frac{1}{n}},$$

which is the Brunn-Minkowski inequality. Hence the conjecture of Costa-Cover ((i) of Proposition 2.2.2) implies the Brunn-Minkowski inequality in the case where one set is convex.

Notice also that the conjecture of Costa-Cover provides a refinement of the Brunn-Minkowski inequality for parallel sets of a fixed compact set A since it then follows that for every $\lambda \in [0, 1]$ and every $t_1, t_2 \in \mathbb{R}_+$,

$$\begin{aligned} |(1 - \lambda)(A + t_1 B) + \lambda(A + t_2 B)|^{\frac{1}{n}} &\geq |A + ((1 - \lambda)t_1 + \lambda t_2)B|^{\frac{1}{n}} \\ &\geq (1 - \lambda)|A + t_1 B|^{\frac{1}{n}} + \lambda|A + t_2 B|^{\frac{1}{n}}. \end{aligned}$$

Let us study the connection with the isoperimetric inequality. The Costa-Cover conjecture implies that for every $t \geq 0$ and every sufficiently regular compact set A

$$\frac{1}{n} \frac{|\partial A|_{n-1}}{|A|^{1-\frac{1}{n}}} = (V_A^{1/n})'_+(0) \geq (V_A^{1/n})'_+(t) \geq \lim_{t \rightarrow +\infty} (V_A^{1/n})'_+(t) = \frac{1}{n} \frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{1-\frac{1}{n}}},$$

which is the isoperimetric inequality. This would give a non-increasing path from $\frac{|\partial A|_{n-1}}{|A|^{1-\frac{1}{n}}}$ to $\frac{|\partial B_2^n|_{n-1}}{|B_2^n|^{1-\frac{1}{n}}}$ through the family

$$\left(\frac{|\partial(A + tB_2^n)|_{n-1}}{|A + tB_2^n|^{1-\frac{1}{n}}} \right)_{t \in \mathbb{R}_+}.$$

We may apply the same arguments for arbitrary convex body B instead of B_2^n . Thus, the conjecture that $t \mapsto V_{A,B}(t)^{1/n}$ is concave on \mathbb{R}_+ implies the following generalized isoperimetric inequality, also known as Minkowski's first inequality proved for example in [77],

$$\frac{|\partial_B A|_{n-1}}{|A|^{1-\frac{1}{n}}} \geq \frac{|\partial_B B|_{n-1}}{|B|^{1-\frac{1}{n}}} = n|B|^{\frac{1}{n}}.$$

Recall that for $t \geq 0$, $V_A(t) = |A + tB_2^n|$ and that Costa and Cover [24] conjectured the $\frac{1}{n}$ -concavity of V_A on \mathbb{R}_+ , for every compact A . They also noticed that their conjecture holds true for A being convex. Let us repeat their argument. For every $\lambda \in [0, 1]$ and t ,

$s \in \mathbb{R}_+$, from the Brunn-Minkowski inequality, one obtains

$$\begin{aligned} |A + ((1 - \lambda)t + \lambda s)B_2^n|^{\frac{1}{n}} &= |(1 - \lambda)(A + tB_2^n) + \lambda(A + sB_2^n)|^{\frac{1}{n}} \\ &\geq (1 - \lambda)|A + tB_2^n|^{\frac{1}{n}} + \lambda|A + sB_2^n|^{\frac{1}{n}}. \end{aligned}$$

Notice that from the same argument we deduce that for every convex sets A and B , the function $V_{A,B}(t) = |A + tB|$ is $\frac{1}{n}$ -concave on \mathbb{R}_+ . Hence for convex sets A and B , the properties (i)-(iv) of Proposition 2.2.2 holds true. In this case, the $\frac{1}{n}$ -concavity of $V_{A,B}$ on \mathbb{R}_+ is equivalent to the Brunn-Minkowski inequality (and true).

2.3 The $\frac{1}{n}$ -concavity of the parallel volume

In dimension 1

Let us prove the Costa-Cover conjecture in dimension 1.

Proposition 2.3.1. *Let A be a compact set in \mathbb{R} and B be a convex body in \mathbb{R} , then $t \mapsto V_{A,B}(t) = |A + tB|$ is concave on \mathbb{R}_+ .*

Proof. We note that in dimension 1, for $t_0 > 0$, $A + t_0B$ is a disjoint finite union of intervals. Thus, by setting $A + t_0B$ for an arbitrary $t_0 > 0$ instead of A , we can assume that $A = \cup_{i=1}^N [a_i, b_i]$, with $a_i, b_i \in \mathbb{R}$, $N \in \mathbb{N}^*$. Thus, for t sufficiently small,

$$V_{A,B}(t) = |A + tB| = \sum_{i=1}^N (b_i - a_i + |B|t) = \sum_{i=1}^N (b_i - a_i) + |B|Nt.$$

Thus $V_{A,B}$ is piecewise affine on \mathbb{R}_+^* . Moreover, when t increases, the slope of $V_{A,B}$ is non-increasing since the number of intervals composing $A + tB$ is non-increasing. Using that $V_{A,B}$ is continuous on \mathbb{R}_+ , we conclude that it is concave on \mathbb{R}_+ . \square

Remark. For arbitrary compact sets A and B , the function $V_{A,B}$ is not necessarily concave as can be seen from the example of $A = \{0; 4\}$ and $B = [-5, -3] \cup [3, 5]$, the same example which was given in the remark after Proposition 2.2.1 to show that the function $V_{A,B}$ is not necessarily increasing.

In dimension 2

We first prove the Costa-Cover conjecture for compact connected sets in dimension 2.

Theorem 2.3.2. *Let A be a compact connected subset of \mathbb{R}^2 . Then, $V_A : t \mapsto |A + tB_2^2|$ is $\frac{1}{2}$ -concave on \mathbb{R}_+ .*

Proof. We proceed by approximating A by finite sets, hence let us first assume that A is finite, $A = \{x_1, \dots, x_N\}$. Let $T = \{t_1, \dots, t_m\} \subset \mathbb{R}_+$, with $t_1 < \dots < t_m$, be the finite set of real numbers which are equal to $\frac{|x_i - x_j|}{2}$ for some $i, j \in \{1, \dots, N\}$ or to the radius of the circumscribed circle of a triangle (x_i, x_j, x_k) for some $i, j, k \in \{1, \dots, N\}$. For $t > 0$, let $p_A(t)$ be the number of connected components of $A + tB_2^2$ and $q_A(t)$ be the genus of $A + tB_2^2$. Notice that the functions p_A and q_A are piecewise constants on $\mathbb{R}_+ \setminus T$ and that V_A is infinitely differentiable on $\mathbb{R}_+ \setminus T$, (see proposition 4.8 in [42]).

We use a key result established by Fiala in the context of Riemannian manifolds, see [32], first part, section 9 *vraies parallèles* (see Remark (\star) below to have an intuition of this result): for every $t \in (0; +\infty) \setminus \{t_1, \dots, t_m\}$,

$$V_A''(t) \leq 2\pi(p_A(t) - q_A(t)).$$

Notice that $p_A(t) - q_A(t)$ is equal to the Euler-Poincaré characteristic of $A + tB_2^2$.

Now, we consider $t_0 \in \mathbb{R}_+$ such that $A + t_0B_2^2$ is connected. Then for every $t \geq t_0$, $A + tB_2^2$ is connected. Hence for every $t \in (t_0; +\infty) \setminus \{t_1, \dots, t_m\}$,

$$V_A''(t) \leq 2\pi. \tag{2.5}$$

Let us prove that V_A is $\frac{1}{2}$ -concave on $(t_0, +\infty)$. By the isoperimetric inequality, we have for every $t \in (t_0; +\infty) \setminus \{t_1, \dots, t_m\}$,

$$4\pi|A + tB_2^2| \leq |\partial(A + tB_2^2)|^2,$$

we write this in this form

$$4\pi V_A(t) \leq V_A'(t)^2,$$

thus, using (2.5),

$$2V_A(t)V_A''(t) \leq V_A'(t)^2.$$

Hence $(\sqrt{V_A})''(t) \leq 0$. We conclude that V_A is $\frac{1}{2}$ -concave on (t_i, t_{i+1}) , for all $i \leq m - 1$ and on $(t_m, +\infty)$. From (2.3) we have $(V_A)'_-(t_i) \geq (V_A)'_+(t_i)$, thus V_A is $\frac{1}{2}$ -concave on $(t_0, +\infty)$.

Let us then consider a compact connected set A of \mathbb{R}^2 . Let $t_0 > 0$. Let $(x_N)_{N \in \mathbb{N}^*}$ be a dense sequence in A . We denote, for $N \in \mathbb{N}^*$, $A_N = \{x_1, \dots, x_N\}$. There exists

$N_0 \in \mathbb{N}^*$ such that for every $N \geq N_0$, $A_N + t_0 B_2^2$ is connected. For every $N \geq N_0$, we have shown that V_{A_N} is $\frac{1}{2}$ -concave on $(t_0; +\infty)$. Moreover the sequence $(A_N)_N \rightarrow A$ in the Hausdorff distance, thus by denoting $d_N = d_H(A_N, A)$, the Hausdorff distance, one has, for every $t > 0$

$$A_N + tB_2^2 \subset A + tB_2^2 \subset A_N + (t + d_N)B_2^2.$$

Applying the right-hand side inclusion to t replaced by $t - d_N$ where N satisfies $d_N < t$, we deduce

$$A + (t - d_N)B_2^2 \subset A_N + tB_2^2 \subset A + tB_2^2.$$

Hence by continuity of the function V_A at the point t ,

$$\lim_{N \rightarrow +\infty} V_{A_N}(t) = V_A(t).$$

It follows that $\sqrt{V_A}$ is the pointwise limit of a sequence of concave functions, hence V_A is $\frac{1}{2}$ -concave on $(t_0; +\infty)$, for every $t_0 > 0$. We conclude that V_A is $\frac{1}{2}$ -concave on \mathbb{R}_+ . \square

Remark (\star). The Dirichlet cells with respect to A (see Figure 2.1 below) are defined for $i \in \{1, \dots, N\}$ by

$$D_i = \{x \in \mathbb{R}^2; |x - x_i| \leq |x - x_j|, \forall j \in \{1, \dots, N\}\}.$$

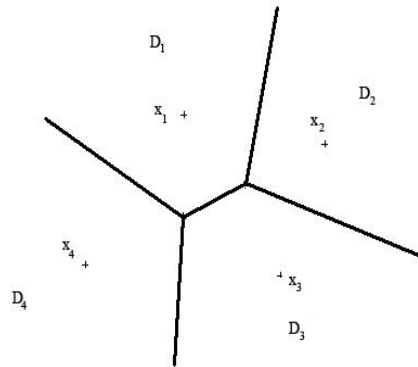


Figure 2.1: Dirichlet cells

For $i \in \{1, \dots, N\}$ and $t \geq 0$, let $V_A^i(t) = |(A + tB_2^2) \cap D_i|$. Notice that

$$V_A(t) = \sum_{i=1}^N V_A^i(t).$$

Let $i \in \{1, \dots, N\}$. We compute $(V_A^i)''(t)$ in the case where D_i is not bounded and $\partial(x_i + tB_2^2) \cap \partial D_i$ is a set of exactly four points u_1, u_2, u_3, u_4 . It will be easy to deduce $(V_A^i)''(t)$ in the other cases. Say for example that the arc $\widehat{u_1 u_2} \subset D_i$ and the arc $\widehat{u_3 u_4} \subset D_i$. Since D_i is a polyhedron, the points u_1, u_2, u_3, u_4 belong to at most four edges, denoted by L_1, L_2, L_3, L_4 . We denote by θ_i the angle $\angle(\text{Aff}(L_1), \text{Aff}(L_2))$, by ν_i the angle $\angle(\text{Aff}(L_3), \text{Aff}(L_4))$, by $\delta_i(t)$ the angle $\angle(u_1, x_i, u_2)$, by $\xi_i(t)$ the angle $\angle(u_3, x_i, u_4)$. The case $L_1 = L_2$ corresponds to $\theta_i = \pi$ and the case L_1 parallel to L_2 corresponds to $\theta_i = 0$, and similarly for ν_i . We denote $a_i = d(x_i, L_1)$ and $b_i = d(x_i, L_2)$. We define the angles $\alpha_i(t)$ and $\beta_i(t)$ such that

$$\sin(\alpha_i(t)) = \frac{a_i}{t} \quad \text{and} \quad \sin(\beta_i(t)) = \frac{b_i}{t}.$$

Say for example that $\delta_i(t) = \theta_i + \alpha_i(t) + \beta_i(t)$. See Figure 2.2 below:

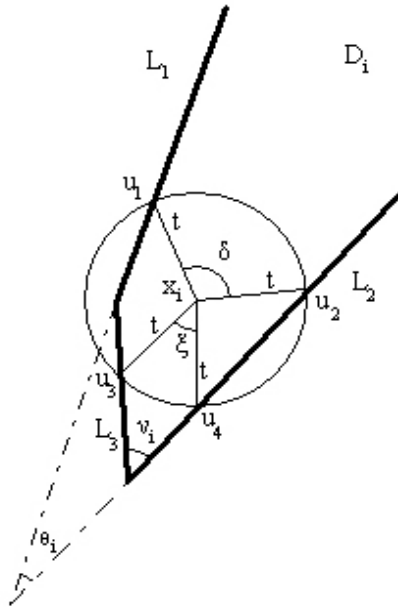


Figure 2.2: Inside a Dirichlet cell

Notice that

$$(V_A^i)'(t) = t\delta_i(t) + t\xi_i(t).$$

We have

$$(t\delta_i(t))' = \theta_i + \alpha_i(t) - \tan(\alpha_i(t)) + \beta_i(t) - \tan(\beta_i(t)) \leq \theta_i.$$

With a similar computation, we also have

$$(t\xi_i(t))' \leq -\nu_i.$$

we conclude that

$$(V_A^i)''(t) \leq \theta_i - \nu_i.$$

Pursuing these computations and adding the obtained inequality, we have

$$V_A''(t) = \sum_{i=1}^N (V_A^i)''(t) \leq 2\pi(p_A(t) - q_A(t)).$$

Remarks.

1. In the proof of Theorem 2.3.2, from the bound $V_A''(t) \leq 2\pi(p_A(t) - q_A(t))$ obtained for every finite set A and for every $t > 0$ outside a finite number of points, one deduces that for every compact subset A of \mathbb{R}^2 with finite connected components p_A , the function $t \mapsto V_A(t) - p_A\pi t^2$ is concave on $(0; +\infty)$. From Steiner's formula one has

$$V_{\text{conv}(A)}(t) = |\text{conv}(A)| + t|\partial(\text{conv}(A))| + \pi t^2.$$

If A is connected, it follows that

$$V_{\text{conv}(A)}(t) - V_A(t) = |\text{conv}(A)| + t|\partial(\text{conv}(A))| + \pi t^2 - V_A(t)$$

is convex as the sum of an affine function and a convex function. Notice that this complements the result of Kampf [53] who proved that $V_{\text{conv}(A)}(t) - V_A(t)$ tends to 0 as $t \rightarrow +\infty$.

2. If in Theorem 2.3.2 we replace B_2^2 by an ellipsoid, *i.e.* by $T(B_2^2)$ where T is an invertible linear transformation, then the result holds since

$$|A + tT(B_2^2)| = |T(T^{-1}(A) + tB_2^2)| = |\det(T)||T^{-1}(A) + tB_2^2|.$$

For a non-connected set A , the next proposition shows that the function V_A is not necessarily $\frac{1}{n}$ -concave on \mathbb{R}_+ in dimension $n \geq 2$.

Proposition 2.3.3. *Let $n \geq 2$. We set $A = B_2^n \cup \{2e_1\}$ (see Figure (2.3) below). The function $V_A(t) = |A + tB_2^n|$ is not $\frac{1}{n}$ -concave on \mathbb{R}_+ .*

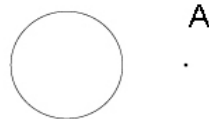


Figure 2.3: Counterexample

Proof. For every $t \in [0, \frac{1}{2})$, we have

$$|A + tB_2^n| = |(B_2^n \cup \{2e_1\}) + tB_2^n| = |B_2^n + tB_2^n| + |tB_2^n| = |B_2^n|((1+t)^n + t^n).$$

Since the $\frac{1}{n}$ -power of this function is not concave (it is strictly convex), V_A is not $\frac{1}{n}$ -concave on \mathbb{R}_+ for $n \geq 2$. □

Remarks.

1. This counterexample shows that the Brunn-Minkowski inequality doesn't imply the $\frac{1}{n}$ -concavity of the parallel volume for non convex sets.
2. The concavity of entropy power in Information theory and the $\frac{1}{n}$ -concavity of the parallel volume in the Brunn-Minkowski theory can be seen as the concavity of the same underlying quantity in Information theory, namely the p -th Rényi entropy power, respectively for $p = 1$ and for $p = 0$. A concavity property for the p -th Rényi entropy power has been shown by Savaré and Toscani in [75] for $p > 1 - \frac{2}{n}$, where n is the dimension and Proposition 2.3.3 shows that it does not hold for $p = 0$.

In dimension $n \geq 3$

We may ask if the Costa-Cover conjecture still holds for connected sets in dimension $n \geq 3$. The next proposition shows that this is false: even for star-shaped body, the function V_A is not necessarily $\frac{1}{n}$ -concave on \mathbb{R}_+ .

Proposition 2.3.4. *Let $n \geq 3$. We set $A = ([-1, 1]^3 \cup [e_1, le_1]) \times [-1, 1]^{n-3}$, where $l \geq n^4$ (see Figure (2.4) below). The function $V_A(t) = |A + tB_2^n|$ is not $\frac{1}{n}$ -concave on \mathbb{R}_+ .*

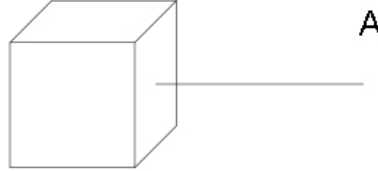


Figure 2.4: Counterexample

Proof. Define $C = \{0\}^3 \times [-1, 1]^{n-3}$. For $t \in [0, 1]$, we have

$$\begin{aligned} |A + tB_2^n| &= |[-1, 1]^n + tB_2^n| + |[(1+t)e_1, le_1] + C + tB_2^n \cap e_1^\perp| \\ &\quad + |\{le_1\} + C + t(B_2^n)^+| \end{aligned}$$

where

$$(B_2^n)^+ = \{x \in B_2^n : x_1 \geq 0\}.$$

We use Steiner's formula for each term. One has

$$|[-1, 1]^n + tB_2^n| = \sum_{k=0}^n \binom{n}{k} t^k 2^{n-k} |B_2^k|_k.$$

For the second term, we first notice that

$$|[(1+t)e_1, le_1] + C + tB_2^n \cap e_1^\perp|_n = (l-1-t)|C + tB_2^n \cap e_1^\perp|_{n-1}.$$

Using for example [77] p. 294 formula (5.3.23), we deduce that the coefficient of t^2 in the Steiner expansion of $|C + tB_2^n \cap e_1^\perp|_{n-1}$ is equal to $\pi 2^{n-3}$. The third term is equal to t^3 times a polynomial. Thus, there are coefficients a_0, \dots, a_n such that for $t \in [0, 1]$,

$$V_A(t) = a_0 + a_1 t + \dots + a_n t^n,$$

with $a_0 = 2^n$, $a_1 = n2^n$ and $a_2 = 2^{n-3}\pi(n(n-1) + l-1)$. Since $l \geq 2n^2$, it follows directly that

$$\frac{n}{n-1} V_A(0) V_A''(0) - V_A'(0)^2 > 0.$$

Hence $(V_A^{1/n})''(0) > 0$, thus $V_A^{1/n}$ is not concave in a neighborhood of 0. □

We have seen that the Costa-Cover conjecture does not hold in general. We still conjecture that the following weaker form may hold.

Conjecture 2.3.5. *Let A be a compact subset of \mathbb{R}^n and B be a convex body in \mathbb{R}^n . Then there exists t_0 such that the function $V_{A,B}(t) = |A + tB|$ is $\frac{1}{n}$ -concave on $[t_0, +\infty)$.*

We have shown that this conjecture is true in dimension 1 and in dimension 2 for $B = B_2^2$. Indeed, in dimension 2, we have seen that it is true for every compact connected set. Since for every compact subset A of \mathbb{R}^2 the set $A + tB_2^2$ is connected for $t \geq \frac{1}{2}\text{diam}(A)$, it follows that $t \mapsto |A + tB_2^2|$ is $\frac{1}{2}$ -concave on $[\frac{1}{2}\text{diam}(A), +\infty)$.

We prove the Conjecture 2.3.5 in some particular cases in dimension $n \geq 3$.

Proposition 2.3.6. *Let A be a compact subset of \mathbb{R}^n . Then the function $t \mapsto |A + t\text{conv}(A)|^{1/n}$ is affine on $[n; +\infty)$. If moreover $\partial\text{conv}(A) \subset A$ then $t \mapsto |A + t\text{conv}(A)|^{1/n}$ is affine on $[1; +\infty)$.*

Proof. It was noticed by Schneider [76] that for every $t \geq n$,

$$A + t\text{conv}(A) = (1 + t)\text{conv}(A).$$

We conclude that $t \mapsto |A + t\text{conv}(A)|^{1/n}$ is affine on $[n; +\infty)$.

If moreover $\partial\text{conv}(A) \subset A$ then for every $x \in \text{conv}(A)$ there exists two points y, z in $\partial\text{conv}(A)$ such that $x \in [y, z]$. Say, for example, that $|x - y| \leq |x - z|$ then $u = 2x - y \in [y, z] \subset \text{conv}(A)$. Hence

$$x = \frac{y + u}{2} \in \frac{\partial\text{conv}(A) + \text{conv}(A)}{2}.$$

Finally

$$\text{conv}(A) \subset \frac{\partial\text{conv}(A) + \text{conv}(A)}{2} \subset \frac{A + \text{conv}(A)}{2} \subset \text{conv}(A).$$

We deduce that $A + t\text{conv}(A) = (1 + t)\text{conv}(A)$, for every $t \geq 1$. We conclude that $t \mapsto |A + t\text{conv}(A)|^{1/n}$ is affine on $[1; +\infty)$. □

Remark. More generally, Schneider introduced in [76] the quantity

$$c(A) = \inf\{t \geq 0; A + t\text{conv}(A) = (1+t)\text{conv}(A)\}.$$

Clearly $t \mapsto |A + t\text{conv}(A)|^{\frac{1}{n}}$ is affine on $[c(A); +\infty)$. The above proposition establishes that $c(A) \leq n$ in general and $c(A) \leq 1$ if $\partial\text{conv}(A) \subset A$. Notice that if $A \subset \mathbb{R}^n$ is connected then $c(A) \leq n - 1$, see [76].

Theorem 2.3.7. *Let A be a compact set in \mathbb{R}^n . If the function $\varepsilon \mapsto |\varepsilon A + B_2^n|$ is twice differentiable in a neighbourhood of 0, with second derivative continuous at 0, then there exists $t_0 \geq 0$ such that the function $V_A(t) = |A + tB_2^n|$ is $\frac{1}{n}$ -concave for $t \geq t_0$. In particular this holds for A being finite.*

Proof. Kampf proved in [54], Lemma 28, that for every compact set A there exists a constant C which depends on n, A so that for every $t \geq 1$,

$$0 \leq |\text{conv}(A) + tB_2^n| - |A + tB_2^n| \leq Ct^{n-3}.$$

Then, setting $\varepsilon = \frac{1}{t}$, for every $\varepsilon \in (0, 1]$, one deduces

$$0 \leq |\varepsilon\text{conv}(A) + B_2^n| - |\varepsilon A + B_2^n| \leq C\varepsilon^3. \tag{2.6}$$

We denote $g_{\text{conv}(A)}(\varepsilon) = |\varepsilon\text{conv}(A) + B_2^n|$ and $g_A(\varepsilon) = |\varepsilon A + B_2^n|$, since g_A is twice differentiable at 0 it follows that

$$g_A(0) = g_{\text{conv}(A)}(0); g'_A(0) = g'_{\text{conv}(A)}(0); g''_A(0) = g''_{\text{conv}(A)}(0).$$

From Steiner's formula, we have $g_{\text{conv}(A)}(0) = |B_2^n|$ and

$$\begin{aligned} g'_{\text{conv}(A)}(0) &= nV(\text{conv}(A), B_2^n[n-1]), \\ g''_{\text{conv}(A)}(0) &= n(n-1)V(\text{conv}(A)[2], B_2^n[n-2]). \end{aligned}$$

If $\text{conv}(A)$ is not homothetic to B_2^n , then from the equality case of the Alexandrov-Fenchel inequality, see [77], Theorem 6.6.8, page 359, we have

$$|B_2^n|V(\text{conv}(A)[2], B_2^n[n-2]) < V(\text{conv}(A), B_2^n[n-1])^2,$$

that is

$$\frac{n}{n-1}g_{\text{conv}(A)}(0)g''_{\text{conv}(A)}(0) < g'_{\text{conv}(A)}(0)^2.$$

Thus we deduce that

$$\frac{n}{n-1}g_A(0)g''_A(0) < g'_A(0)^2.$$

Since g_A , g'_A and g''_A are continuous at 0, there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in [0, \varepsilon_0]$,

$$\frac{n}{n-1}g_A(\varepsilon)g''_A(\varepsilon) \leq g'_A(\varepsilon)^2.$$

Hence the function g_A is $\frac{1}{n}$ -concave on $[0, \varepsilon_0]$. We conclude by Proposition 2.2.2, setting $t_0 = \frac{1}{\varepsilon_0}$, that $t \mapsto |A + tB_2^n|$ is $\frac{1}{n}$ -concave on $[t_0, +\infty)$. If $\text{conv}(A)$ is homothetic to B_2^n then the result follows from Proposition 2.3.6.

If A is finite then the function $\varepsilon \mapsto |\varepsilon A + B_2^n|$ is analytic in a neighbourhood of 0, see [42].

□

Remarks.

1. The preceding theorem is still valid if one replaces B_2^n by a convex body $B = rB_2^n + M$, for some $r > 0$ and some convex body M such that its support function $h_B(u) = \max\{\langle x, u \rangle, x \in B\}$ is twice differentiable on $\mathbb{R}^n \setminus \{0\}$ because inequality (2.6) holds with these assumptions (see [54]).
2. The function $\varepsilon \mapsto |\varepsilon A + B_2^n|$ is not necessarily twice differentiable in a neighbourhood of 0 as can be seen from the following example. In dimension 2, we consider the points $I = (1, 1)$, $J = (1, 0)$ and $A = I \cup J \cup \{(\cos(1/k), \sin(1/k)), k \geq 1\}$. Then, A is compact but for every $t_0 \in \mathbb{R}_+$, the function $V_A(t) = |A + tB_2^2|$ is not twice differentiable on $(t_0, +\infty)$.

In fact, one can show that the function $V_A(t) = |A + tB_2^n|$ is continuously differentiable on $[\text{diam}(A); +\infty)$.

Proposition 2.3.8. *Let A be a compact subset of \mathbb{R}^n . Then the function $V_A(t) = |A + tB_2^n|$ is continuously differentiable on $[\text{diam}(A); +\infty)$, the function $g_A(\varepsilon) = |\varepsilon A + B_2^n|$ is continuously differentiable on $(0, \text{diam}(A)^{-1}]$ and differentiable at 0 with $g'_A(0) = nV(\text{conv}(A), B_2^n[n-1])$.*

Proof. Rataj et al. in [72], Theorem 3.3, showed that $V'_A(t)$ exists for every $t \geq \text{diam}(A)$, thus we have for every $t \geq \text{diam}(A)$

$$V'_A(t) = |\partial(A + tB_2^n)|.$$

Moreover, if (A_N) is a sequence of non-empty compact subset of \mathbb{R}^n tending in Hausdorff distance to a compact subset A of \mathbb{R}^n , then by [79], theorem 3, for every $t > 0$ such that $V'_A(t)$ exists

$$\lim_{N \rightarrow +\infty} |\partial(A_N + tB_2^n)| = |\partial(A + tB_2^n)|.$$

Let $t \geq \text{diam}(A)$, we apply this result to $A_N = A + t_N B_2^n$, where (t_N) is a sequence of non-negative numbers tending to 0. We obtain that

$$\lim_{N \rightarrow +\infty} V'_A(t + t_N) = V'_A(t).$$

Hence, V'_A is right continuous at t . Let t, t_0 be such that $t > t_0 > \text{diam}(A)$, we now apply the result of Stachó to $A_N = A + (t_0 - t_N)B_2^n$, where (t_N) is a sequence of non-negative numbers tending to 0. We obtain

$$\lim_{N \rightarrow +\infty} |\partial(A_N + (t - t_0)B_2^n)| = |\partial(A + t_0 B_2^n + (t - t_0)B_2^n)|$$

that is

$$\lim_{N \rightarrow +\infty} V'_A(t - t_N) = V'_A(t).$$

Hence, V'_A is left continuous at t . We conclude that V_A is continuously differentiable on $[\text{diam}(A), +\infty)$.

Let us denote $g_A(\varepsilon) = |\varepsilon A + B_2^n|$. Since

$$g_A(\varepsilon) = |\varepsilon A + B_2^n| = \varepsilon^n V_A\left(\frac{1}{\varepsilon}\right)$$

one deduces that g_A is continuously differentiable on $(0, \frac{1}{\text{diam}(A)}]$. Moreover, from the inequality (2.6), valid for every compact set A , one deduces that g_A is also differentiable at 0, with $g'_A(0) = nV(\text{conv}(A), B_2^n[n-1])$. \square

A special case in dimension 3

We have seen that for every finite subset A of \mathbb{R}^n , there exists $t_0(A)$ such that the function $V_A(t) = |A + tB_2^n|$ is $\frac{1}{n}$ -concave for $t \geq t_0(A)$. In dimension 3, we can give a

bound on $t_0(A)$ in terms of the geometry of A .

In the sequel, A denotes a finite subset of \mathbb{R}^3 . We denote by D_i a Dirichlet-Voronoi cell with respect to $A = \{x_1, \dots, x_N\}$, defined for $i \in \{1, \dots, N\}$ by

$$D_i = \{x \in \mathbb{R}^3; |x - x_i| \leq |x - x_j|, \forall j \in \{1, \dots, N\}\}.$$

The following condition can be found in [55].

Condition (\star) For all faces F of the polytope $\text{conv}(A)$, and all edges E of F , we have

$$\forall x \in E, \quad d(x, A \cap E) = d(x, A \cap F).$$

For example, if $\text{conv}(A)$ is simplicial, this condition holds if and only if each face of $\text{conv}(A)$ is a triangle with only acute angles. In general, this condition holds if and only if for every face F of $\text{conv}(A)$, for every edge $[a, b]$ of F and for every vertex c of F , the angle (ca, cb) is acute.

Proposition 2.3.9. *Let A be a finite set in \mathbb{R}^3 satisfying the condition (\star) . Then, $V_A(t) = |A + tB_2^3|$ is $\frac{1}{3}$ -concave on $[t_0(A); +\infty)$, where*

$$t_0(A) = \min\{t \geq \text{diam}(A); D_i \subset A + tB_2^3, \text{ for all bounded } D_i\}.$$

Proof. Kampf and Kiderlen have shown in [55] that for every $t > t_0(A)$,

$$|\text{conv}(A) + tB_2^3| - |A + tB_2^3| = a_0 + \sum_{p \geq 1} a_p t^{-2p+1}$$

with for all $p \geq 0$, $a_p \geq 0$. Since $V_{\text{conv}(A)}$ is polynomial thus V_A is twice differentiable on $(t_0(A); +\infty)$. It follows that for every $t > t_0(A)$,

$$\begin{aligned} V'_A(t) &= V'_{\text{conv}(A)}(t) + \sum_{p \geq 1} (2p - 1)a_p t^{-2p} \\ V''_A(t) &= V''_{\text{conv}(A)}(t) - \sum_{p \geq 1} 2p(2p - 1)a_p t^{-2p-1}. \end{aligned}$$

Then, for every $t > t_0(A)$,

$$V_A(t) \leq V_{\text{conv}(A)}(t), \quad V'_A(t) \geq V'_{\text{conv}(A)}(t) \quad \text{and} \quad V''_A(t) \leq V''_{\text{conv}(A)}(t). \quad (2.7)$$

The Brunn-Minkowski inequality implies that $V_{\text{conv}(A)}$ is $\frac{1}{3}$ -concave on \mathbb{R}_+ . We conclude that for every $t > t_0(A)$,

$$\frac{3}{2}V_A(t)V_A''(t) \leq \frac{3}{2}V_{\text{conv}(A)}(t)V_{\text{conv}(A)}''(t) \leq V_{\text{conv}(A)}'(t)^2 \leq V_A'(t)^2.$$

So, V_A is $\frac{1}{3}$ -concave on $[t_0(A); +\infty)$. □

Remarks

1. For an arbitrary compact subset A of \mathbb{R}^3 , if there exists a sequence $(x_N)_{N \in \mathbb{N}^*}$ dense in A such that for every N , the set A_N satisfies the condition (\star) , where $A_N = \{x_1, \dots, x_N\}$, and such that $t_0(A_N)$ is uniformly bounded in N by a t_0 , then the function $t \mapsto |A + tB_2^3|$ will be $\frac{1}{3}$ -concave on $[t_0; +\infty)$.
2. In dimension $n \geq 4$, there is no hope to prove the inequalities (2.7) because for A being two points at distance 2, one has for every $t \geq 1$

$$\begin{aligned} V_A'(t) &= n|B_2^n|t^{n-1} + 2(n-1)|B_2^{n-1}|t \int_0^1 (t^2 - x^2)^{\frac{n-3}{2}} dx \\ &< n|B_2^n|t^{n-1} + 2(n-1)|B_2^{n-1}|t^{n-2} = V_{\text{conv}(A)}'(t). \end{aligned}$$

2.4 Further analogies

In Information theory, the Blachman-Stam inequality ([7] and [80]) (1.22), which states that for every independent random vectors X and Y in \mathbb{R}^n with non-zero Fisher information one has

$$\frac{1}{I(X+Y)} \geq \frac{1}{I(X)} + \frac{1}{I(Y)},$$

directly implies all previous mentioned inequalities of Information theory: the entropy power inequality (1.20) (thus the Log-Sobolev inequality for Gaussian measure) and the concavity of entropy power. This last inequality also called the *isoperimetric information inequality* may be deduced from the Blachman-Stam inequality in the same way as the *isoperimetric entropy inequality* was deduced from the entropy power inequality, by applying it to $Y = \sqrt{\varepsilon}G$ and letting ε tend to 0 (see Chapter 1).

Let us now investigate the analogue of the Fisher information and the Blachman-Stam inequality in the Brunn-Minkowski theory. Recall de Bruijn's identity

$$I(X) = \frac{d}{dt}\bigg|_{t=0} 2H(X + \sqrt{t}G).$$

Since the entropy H is the analogue of the logarithm of the volume $\log |\cdot|$, Dembo, Cover and Thomas [29] proposed, as an analogue of the Fisher information I , the quantity

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\log |A + \varepsilon B_2^n|) = \frac{|\partial A|}{|A|},$$

for sufficiently regular compact sets A . Thus, in analogy with the Blachman-Stam inequality, one may wonder if for every regular compact sets A and B ,

$$\frac{|A + B|}{|\partial(A + B)|} \geq \frac{|A|}{|\partial A|} + \frac{|B|}{|\partial B|}. \tag{2.8}$$

Even restricted to the case where A and B are convex sets, checking the validity of this inequality is not an easy task and it was conjectured by Dembo, Cover and Thomas [29] that the inequality (2.8) holds true in this particular case. In [34], it was shown that this conjecture (for convex sets) holds true in dimension 2 but is false in dimension $n \geq 3$. In particular, it was proved that, if $n \geq 3$, there exists a convex body K such that the inequality (2.8) cannot be true for all $A, B \in \{K + tB_2^n; t \geq 0\}$. It was also proved that if B is a segment then there exists a convex body A for which (2.8) is false.

In another direction, one may also ask if (2.8) holds true for B being arbitrary Euclidean ball and for every compact set A . In this case, applying (2.8) to A replaced by $A + sB_2^n$ and $B = (t - s)B_2^n$, one would have, for every $0 \leq s \leq t$,

$$\frac{|A + tB_2^n|}{|\partial(A + tB_2^n)|} \geq \frac{|A + sB_2^n|}{|\partial(A + sB_2^n)|} + (t - s) \frac{|B_2^n|}{|\partial B_2^n|} = \frac{|A + sB_2^n|}{|\partial(A + sB_2^n)|} + \frac{t - s}{n},$$

with the notations given above, this would mean that

$$t \mapsto \frac{V_A(t)}{(V_A)'_+(t)} - \frac{t}{n}$$

is non-decreasing on $(0, +\infty)$. This is equivalent to the $\frac{1}{n}$ -concavity of V_A , which is the Costa-Cover conjecture.

Chapter 3

A functional version of the parallel volume

In this chapter, I present the second part of my paper *Concavity properties of extensions of the parallel volume* [64], published in *Mathematika*. In this paper, I give a functional version of the conjecture of Costa-Cover and I establish concavity properties of this new function.

3.1 Functional version

We have seen in Chapter 1 that geometric inequalities can be extended into functional forms. The most famous extension of this type in the Brunn-Minkowski theory is certainly the Prékopa-Leindler inequality (1.8). Functional versions provide new proofs of geometric inequalities and provide new applications. Other examples of such extensions are a functional version of the Blaschke-Santalò inequality and the Mahler conjecture (see *e.g.* [4], [3], [38], [39], [59]).

In this chapter, we consider a functional version of the parallel sets of $A \subset \mathbb{R}^n$ and then we study concavity properties of a particular function associated with it. More precisely, let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a measurable non-negative function and $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a log-concave function, we define the function $h_t^{(0)}$, $t \in \mathbb{R}_+$, by

$$h_t^{(0)}(z) = \sup_{z=x+ty} f(x)g(y)^t, \quad z \in \mathbb{R}^n.$$

In the following, we study concavity properties of the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(0)}(z) dz$ on \mathbb{R}_+ .

The classical parallel set of A is a particular case. Indeed, by taking $f = 1_A$ and $g = 1_{B_2^n}$, one then has

$$1_A(x)1_{B_2^n}(y) = \begin{cases} 1 & \text{if } x \in A \text{ and } y \in B_2^n \\ 0 & \text{otherwise} \end{cases},$$

hence for every $z \in \mathbb{R}^n$ and for every $t \in \mathbb{R}_+$, $h_t^{(0)}(z) = 1_{A+tB_2^n}(z)$.

We also study concavity properties of the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$ on \mathbb{R}_+ , where

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; g(y)>0}} (f(x)^\gamma + tg(y)^\gamma)^{\frac{1}{\gamma}},$$

and where $\gamma \in \mathbb{R} \setminus \{0\}$, $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a measurable non-negative function and $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a γ -concave function.

In Chapter 2, we have seen that the classical parallel volume of convex sets is $\frac{1}{n}$ -concave on \mathbb{R}_+ . Since γ -concave functions are the analogue of convex sets, it is natural to expect a concavity property of a functional version of the parallel volume for γ -concave functions.

Proposition 3.1.1. *Let $\gamma \geq -\frac{1}{n}$. Let $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be two γ -concave functions. Then the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$ is $\frac{\gamma}{1+\gamma n}$ -concave on \mathbb{R}_+ , where*

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; g(y)>0}} (f(x)^\gamma + tg(y)^\gamma)^{\frac{1}{\gamma}}, \quad h_t^{(0)}(z) = \sup_{z=x+ty} f(x)g(y)^t.$$

Proof. We examine the case $\gamma \neq 0$. The case $\gamma = 0$ can be proved with the same argument. For convenience, let us define $h_t = h_t^{(\gamma)}$. Let $\lambda \in [0, 1]$ and let $t_1, t_2 \in \mathbb{R}_+$. We want to show that

$$\int_{\mathbb{R}^n} h_{(1-\lambda)t_1+\lambda t_2}(z) dz \geq \left((1-\lambda) \left(\int_{\mathbb{R}^n} h_{t_1}(z) dz \right)^{\frac{\gamma}{1+\gamma n}} + \lambda \left(\int_{\mathbb{R}^n} h_{t_2}(z) dz \right)^{\frac{\gamma}{1+\gamma n}} \right)^{\frac{1+\gamma n}{\gamma}}.$$

From the Borell-Brascamp-Lieb inequality, *c.f.* Theorem 1.4.1 (dimensional Prékopa's inequality), it is sufficient to show that

$$\forall z_1, z_2 \in \mathbb{R}^n, h_{(1-\lambda)t_1+\lambda t_2}((1-\lambda)z_1 + \lambda z_2) \geq ((1-\lambda)h_{t_1}(z_1)^\gamma + \lambda h_{t_2}(z_2)^\gamma)^{\frac{1}{\gamma}}.$$

By assumptions, the functions f and g are γ -concave and thus are continuous. By truncation, one can assume that f and g are compactly supported. Hence, the supremum in the definition of h_t is a maximum. Let $z_1, z_2 \in \mathbb{R}^n$. Let $x_1, x_2 \in \mathbb{R}^n$ such that

$$\forall i \in \{1, 2\}, h_{t_i}(z_i) = \left(f(x_i)^\gamma + t_i g\left(\frac{z_i - x_i}{t_i}\right)^\gamma \right)^{\frac{1}{\gamma}}.$$

Let us define $h = h_{(1-\lambda)t_1 + \lambda t_2}((1-\lambda)z_1 + \lambda z_2)$ and $t = (1-\lambda)t_1 + \lambda t_2$. We have

$$\begin{aligned} h &= \sup_{x \in \mathbb{R}^n} \left(f(x)^\gamma + t g\left(\frac{(1-\lambda)z_1 + \lambda z_2 - x}{t}\right)^\gamma \right)^{\frac{1}{\gamma}} \\ &\geq \left(f((1-\lambda)x_1 + \lambda x_2)^\gamma + t g\left(\frac{(1-\lambda)z_1 + \lambda z_2 - ((1-\lambda)x_1 + \lambda x_2)}{t}\right)^\gamma \right)^{\frac{1}{\gamma}} \\ &= \left(f((1-\lambda)x_1 + \lambda x_2)^\gamma + t g\left(\frac{(1-\lambda)t_1}{t} \frac{z_1 - x_1}{t_1} + \frac{\lambda t_2}{t} \frac{z_2 - x_2}{t_2}\right)^\gamma \right)^{\frac{1}{\gamma}} \\ &\geq \left((1-\lambda)f(x_1)^\gamma + \lambda f(x_2)^\gamma + (1-\lambda)t_1 g\left(\frac{z_1 - x_1}{t_1}\right)^\gamma + \lambda t_2 g\left(\frac{z_2 - x_2}{t_2}\right)^\gamma \right)^{\frac{1}{\gamma}} \\ &= ((1-\lambda)h_{t_1}(z_1)^\gamma + \lambda h_{t_2}(z_2)^\gamma)^{\frac{1}{\gamma}}. \end{aligned}$$

□

As a consequence of the Hölder inequality (1.14) (see Corollary 1.4.8), if $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is β -concave and $g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is γ -concave, then fg is α -concave for every $\alpha, \beta, \gamma \in \mathbb{R} \cup \{+\infty\}$ such that $\beta + \gamma \geq 0$ and $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$. A generalized form of Proposition 3.1.1 follows:

Proposition 3.1.2. *Let $\gamma \geq -\frac{1}{n}$. If a measure μ has a β -concave density, with $\beta \geq -\gamma$, and if $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are two γ -concave functions, then $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) d\mu(z)$ is $\frac{\alpha}{1+\alpha n}$ -concave on \mathbb{R}_+ , where $\frac{1}{\beta} + \frac{1}{\gamma} = \frac{1}{\alpha}$.*

Proof. By assumption, $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_+$ are γ -concave functions, with $\gamma \geq -\frac{1}{n}$, and $d\mu(z) = \phi(z) dz$, where $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a β -concave function, with β such that $\beta + \gamma \geq 0$. Let $t_1, t_2 \in \mathbb{R}_+$ and $z_1, z_2 \in \mathbb{R}^n$. We have seen in Proposition 3.1.1 that

$$h_{(1-\lambda)t_1 + \lambda t_2}^{(\gamma)}((1-\lambda)z_1 + \lambda z_2) \geq \left((1-\lambda)h_{t_1}^{(\gamma)}(z_1)^\gamma + \lambda h_{t_2}^{(\gamma)}(z_2)^\gamma \right)^{\frac{1}{\gamma}}.$$

It follows from the Hölder inequality (1.14) (see Corollary 1.4.8) that

$$h_{(1-\lambda)t_1+\lambda t_2}^{(\gamma)}((1-\lambda)z_1+\lambda z_2)\phi((1-\lambda)z_1+\lambda z_2) \geq \left((1-\lambda)(h_{t_1}^{(\gamma)}(z_1)\phi(z_1))^\alpha + \lambda(h_{t_2}^{(\gamma)}(z_2)\phi(z_2))^\alpha \right)^{\frac{1}{\alpha}},$$

where α is such that $\frac{1}{\alpha} = \frac{1}{\beta} + \frac{1}{\gamma}$. From the Borell-Brascamp-Lieb inequality, *c.f.* Theorem 1.4.1, it follows that

$$\int_{\mathbb{R}^n} h_{(1-\lambda)t_1+\lambda t_2}^{(\gamma)}(z)\phi(z) dz \geq \left((1-\lambda) \left(\int_{\mathbb{R}^n} h_{t_1}^{(\gamma)}(z)\phi(z) dz \right)^s + \lambda \left(\int_{\mathbb{R}^n} h_{t_2}^{(\gamma)}(z)\phi(z) dz \right)^s \right)^{\frac{1}{s}},$$

where $s = \frac{\alpha}{1+\alpha n}$. We conclude that the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) d\mu(z)$ is $\frac{\alpha}{1+\alpha n}$ -concave on \mathbb{R}_+ . \square

Before starting the study of concavity properties of the function $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz$ in dimension 1, let us link $h_t^{(\gamma)}$ with the Hopf-Lax solution of the Hamilton-Jacobi equation. Since by assumption g is γ -concave, hence for $\gamma < 0$ one has $g^\gamma = V$, where V is a convex function. It follows that

$$h_t^{(\gamma)}(z) = \sup_{x \in \mathbb{R}^n} \left(f(x)^\gamma + tV \left(\frac{z-x}{t} \right) \right)^{\frac{1}{\gamma}} = \left(Q_t^{(V)} f^\gamma(z) \right)^{\frac{1}{\gamma}},$$

where for arbitrary convex function V and arbitrary function u ,

$$Q_t^{(V)} u(z) = \inf_{x \in \mathbb{R}^n} \left(u(x) + tV \left(\frac{z-x}{t} \right) \right).$$

This expression can be extended to $\gamma = 0$ in the same way, by writing

$$h_t^{(0)}(z) = \sup_{x \in \mathbb{R}^n} f(x) e^{-tV \left(\frac{z-x}{t} \right)} = e^{-Q_t^{(V)}(-\log(f))(z)}.$$

The Hopf-Lax solution is of interest. For example, it can be used to show that the *hypercontractivity* of this solution is equivalent to the *log-Sobolev inequalities* (see *e.g.* [9], [43]). We pursue the study of this solution by investigating concavity properties with respect to the time variable of the Hopf-Lax solution of the Hamilton-Jacobi equation.

Let us recall the basic properties of the Hopf-Lax solution of the Hamilton-Jacobi equation. For a convex function V such that $\lim_{|z| \rightarrow +\infty} V(z)/|z| = +\infty$ and for a Lipschitz function u , it is known that $Q_t^{(V)} u$ is the solution, called the *Hopf-Lax solution*, of the following partial differential equation, called the *Hamilton-Jacobi equation* (see *e.g.*

[30]).

$$\begin{cases} \frac{\partial}{\partial t} h(t, z) + V^*(\nabla h(t, z)) = 0 & \text{on } (0, +\infty) \times \mathbb{R}^n \\ h(t, z) = u(z) & \text{on } \{t = 0\} \times \mathbb{R}^n \end{cases},$$

where V^* is the Legendre transform of V defined on \mathbb{R}^n by

$$V^*(y) = \sup_{x \in \mathbb{R}^n} (\langle x, y \rangle - V(x)).$$

It is shown in [30] that if u is Lipschitz on \mathbb{R}^n then $Q_t^{(V)}u$ is Lipschitz on $[0, +\infty) \times \mathbb{R}^n$. However, for arbitrary convex function V , $t \mapsto Q_t^{(V)}u$ is not necessarily continuous at 0.

Let us show partial positive answers to concavity properties of the function $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz$ in dimension 1. In fact, in dimension 1, we can improve the concavity expected (when restricted to log-concave functions of the form $|y|^p/p, p \geq 1$).

Theorem 3.1.3. *Let $\gamma \in [-1, 0]$. Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$ be such that f^γ (to be interpreted as $-\log(f)$ when $\gamma = 0$) is a bounded Lipschitz function. Define for every $y \in \mathbb{R}$, $V(y) = \frac{|y|^p}{p}$ with $p \geq 1$, and*

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; V(y)>0}} (f(x)^\gamma + tV(y))^\frac{1}{\gamma}, \quad h_t^{(0)}(z) = \sup_{z=x+ty} f(x)e^{-tV(y)}.$$

If for almost every $z \in \mathbb{R}$, $t \mapsto h_t^{(\gamma)}(z)$ is twice differentiable on \mathbb{R}_+ with derivatives that are bounded by an integrable function, and if $\lim_{z \rightarrow \pm\infty} \frac{\partial h_t^{(\gamma)}}{\partial z}(z) = 0$, then the function $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz$ is concave on \mathbb{R}_+ .

Proof. We examine the case $\gamma \neq 0$, the case $\gamma = 0$ can be proved with the same argument. Since $\gamma \in [-1, 0)$, one has

$$h_t^{(\gamma)}(z) = \sup_{x \in \mathbb{R}} \left(f(x)^\gamma + tV\left(\frac{z-x}{t}\right) \right)^\frac{1}{\gamma} = \left(Q_t^{(V)} f^\gamma(z) \right)^\frac{1}{\gamma}.$$

We denote for $t \in \mathbb{R}_+$,

$$F(t) = \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz = \int_{\mathbb{R}} \left(Q_t^{(V)} f^\gamma(z) \right)^\frac{1}{\gamma} dz.$$

For $p = 1$, the function F is constant. We then consider $p > 1$. For convenience, we set $\phi(t, z) = Q_t^{(V)} f^\gamma(z)$ and $\phi' = \frac{\partial \phi}{\partial z}$. By assumption, for almost every $z \in \mathbb{R}$, $t \mapsto h_t^{(\gamma)}(z)$ is twice differentiable on \mathbb{R}_+ with derivatives that are bounded by an integrable function,

hence for every $t \in \mathbb{R}_+$,

$$F'(t) = -\frac{1}{\gamma} \int_{\mathbb{R}} V^*(\phi') \phi^{\frac{1-\gamma}{\gamma}},$$

$$F''(t) = \frac{1}{\gamma} \int_{\mathbb{R}} \phi'' ((V^*)'(\phi'))^2 \phi^{\frac{1-\gamma}{\gamma}} + \frac{1-\gamma}{\gamma^2} \int_{\mathbb{R}} (V^*(\phi'))^2 \phi^{\frac{1-2\gamma}{\gamma}}.$$

We assumed that $V(u) = \frac{|u|^p}{p}$. Hence $V^*(u) = \frac{|u|^q}{q}$, with $\frac{1}{p} + \frac{1}{q} = 1$. It follows that

$$F''(t) = \frac{1}{\gamma} \int_{\mathbb{R}} \phi'' (\phi')^{2q-2} \phi^{\frac{1-\gamma}{\gamma}} + \frac{1-\gamma}{\gamma^2} \int_{\mathbb{R}} \frac{(\phi')^{2q}}{q^2} \phi^{\frac{1-2\gamma}{\gamma}}.$$

By assumption, $\lim_{z \rightarrow \pm\infty} \frac{\partial h_t^{(\gamma)}}{\partial z}(z) = 0$, thus integration by parts gives

$$\frac{2q-1}{\gamma} \int_{\mathbb{R}} \phi'' (\phi')^{2q-2} \phi^{\frac{1-\gamma}{\gamma}} = -\frac{1-\gamma}{\gamma^2} \int_{\mathbb{R}} (\phi')^{2q} \phi^{\frac{1-2\gamma}{\gamma}}.$$

Finally,

$$F''(t) = -\frac{1-\gamma}{\gamma^2} \frac{(q-1)^2}{q^2(2q-1)} \int_{\mathbb{R}} (\phi')^{2q} \phi^{\frac{1-2\gamma}{\gamma}} \leq 0.$$

We conclude that $t \mapsto \int_{\mathbb{R}} h_t^{(\gamma)}(z) dz$ is concave on \mathbb{R}_+ . □

3.2 Links with weighted Brascamp-Lieb-type inequalities

In this section, we express the s -concavity, $s \in \mathbb{R}$, of the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$ in term of a weighted Brascamp-Lieb-type inequality.

Proposition 3.2.1. *Let $\gamma \in [-\frac{1}{n}, 0)$ and let $s \in \mathbb{R}$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be such that f^γ is a bounded Lipschitz function. Let $V : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a convex function such that $\lim_{|z| \rightarrow +\infty} V(z)/|z| = +\infty$. Let us define for every $z \in \mathbb{R}^n$,*

$$h_t^{(\gamma)}(z) = \sup_{\substack{z=x+ty \\ f(x)>0; V(y)>0}} (f(x)^\gamma + tV(y))^{\frac{1}{\gamma}}.$$

If there exists $t_0 > 0$ such that for almost every $z \in \mathbb{R}^n$, $t \mapsto h_t^{(\gamma)}(z)$ is twice differentiable on $[0, t_0]$ with derivatives that are bounded by an integrable function and if $G \in L^2(\mu)$, where $d\mu(z) = \left((Q_t^{(V)} f^\gamma(z))^{\frac{1}{\gamma}} / \int (Q_t^{(V)} f^\gamma)^{\frac{1}{\gamma}} \right) dz$ and $G = V^(\nabla_z Q_t^{(V)} f^\gamma) / Q_t^{(V)} f^\gamma$, then the function $t \mapsto \int_{\mathbb{R}^n} h_t^{(\gamma)}(z) dz$ is s -concave on $[0, t_0]$ if and only if*

$$\begin{aligned} \text{Var}_\mu(G) \leq & -\frac{\gamma}{1-\gamma} \int \frac{\langle (\text{Hess } Q_t^{(V)} f^\gamma)(\nabla_z V^*)(\nabla_z Q_t^{(V)} f^\gamma), (\nabla_z V^*)(\nabla_z Q_t^{(V)} f^\gamma) \rangle}{Q_t^{(V)} f^\gamma} d\mu \\ & + \frac{\gamma-s}{1-\gamma} \left(\int G d\mu \right)^2, \end{aligned}$$

Proof. Recall that $h_t^{(\gamma)}(z) = \left(Q_t^{(V)} f^\gamma(z) \right)^{\frac{1}{\gamma}}$, where for arbitrary convex function V and arbitrary function u ,

$$Q_t^{(V)} u(z) = \inf_{x \in \mathbb{R}^n} \left(u(x) + tV \left(\frac{z-x}{t} \right) \right).$$

For convenience, let us denote $\phi = f^\gamma$ and $Q_t = Q_t^{(V)}$. We have

$$\frac{\partial h_t^{(\gamma)}}{\partial t} = -\frac{1}{\gamma} V^*(\nabla_z Q_t \phi) (Q_t \phi)^{\frac{1-\gamma}{\gamma}},$$

$$\begin{aligned} \frac{\partial^2 h_t^{(\gamma)}}{\partial t^2} &= \frac{1}{\gamma} \langle (\text{Hess } Q_t \phi)(\nabla_z V^*)(\nabla_z Q_t \phi), (\nabla_z V^*)(\nabla_z Q_t \phi) \rangle (Q_t \phi)^{\frac{1-\gamma}{\gamma}} \\ &\quad + \frac{1-\gamma}{\gamma^2} (V^*(\nabla_z Q_t \phi))^2 (Q_t \phi)^{\frac{1-2\gamma}{\gamma}}. \end{aligned}$$

Thus the function F is s -concave if and only if $F(t)F''(t) \leq (1-s)F'(t)^2$ if and only if

$$\begin{aligned} \text{Var}_\mu(G) &\leq -\frac{\gamma}{1-\gamma} \int \frac{\langle (\text{Hess } Q_t \phi)(\nabla_z V^*)(\nabla_z Q_t \phi), (\nabla_z V^*)(\nabla_z Q_t \phi) \rangle}{Q_t \phi} d\mu \\ &\quad + \frac{\gamma-s}{1-\gamma} \left(\int G d\mu \right)^2, \end{aligned}$$

where $d\mu(z) = \left((Q_t \phi(z))^{\frac{1}{\gamma}} / \int (Q_t \phi)^{\frac{1}{\gamma}} \right) dz$ and $G = (V^*(\nabla_z Q_t \phi) / Q_t \phi)$. \square

Remark. For $\gamma = 0$ and $V(u) = \frac{|u|^2}{2}$, one may use the same argument to see that $t \mapsto \int_{\mathbb{R}^n} h_t^{(0)}(z) dz$ is log-concave if and only if

$$\text{Var}_\mu(|\nabla_z Q_t \phi|^2) \leq 4 \int \langle (\text{Hess } Q_t \phi) \nabla_z Q_t \phi, \nabla_z Q_t \phi \rangle d\mu,$$

where $\phi = -\log f$ and $d\mu(z) = (e^{-Q_t \phi(z)} / \int e^{-Q_t \phi}) dz$.

From Proposition 3.1.1, if f is γ -concave with $\gamma \in [-\frac{1}{n}, 0)$, then one may apply Proposition 3.2.1 to obtain the following weighted Brascamp-Lieb-type inequality by letting t go to 0.

Corollary 3.2.2. *Let $\gamma \in [-\frac{1}{n}, 0)$ and let $s = \frac{\gamma}{1+\gamma n}$. For every $V, \phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ convex such that $\lim_{|z| \rightarrow +\infty} V(z)/z = +\infty$, one has*

$$\text{Var}_\mu(G) \leq -\frac{\gamma}{1-\gamma} \int \frac{\langle (\text{Hess } \phi)^{-1} \nabla_z G \phi, \nabla_z G \phi \rangle}{\phi} d\mu + \frac{\gamma-s}{1-\gamma} \left(\int G d\mu \right)^2, \quad (3.1)$$

where $d\mu(z) = \left(\phi^{\frac{1}{\gamma}}(z) / \int \phi^{\frac{1}{\gamma}} \right) dz$ and $G = V(\nabla_z \phi) / \phi$.

We have reproved Bobkov-Ledoux's result [11] (for a smaller class of functions G) who used the same idea since inequality (3.1) is derived from the Borell-Brascamp-Lieb inequality (dimensional Prékopa's inequality). Bobkov and Ledoux have already seen in [10] that one can deduce the classical Brascamp-Lieb inequality from the classical Prékopa inequality (corresponding to the log-concave case). This idea has been explored by Cordero-Erausquin and Klartag in [22] where they showed that the converse is true,

i.e. one can derive the Prékopa inequality from the Brascamp-Lieb inequality. Thereafter, Nguyen [68] generalized the work by Cordero-Erausquin and Klartag to the case of s -concave measures (even for $s \geq 0$) and improved Bobkov-Ledoux's Brascamp-Lieb-type inequality (inequality (3.1)). Recently, Kolesnikov and Milman [57] generalized the weighted Brascamp-Lieb-type inequalities obtained by Bobkov, Ledoux and Nguyen to the setting of Riemannian manifolds.

Chapter 4

Concavity properties of convex measures

In this chapter, the first section is devoted to the first part of my paper *Concavity properties of extensions of the parallel volume* [64]. In this paper, I extend the notion of parallel volume by considering convex measures and I prove concavity properties for this new function.

In the second part of this chapter, I present my third paper *On the improvement of concavity of convex measures* [65], published in Proceedings of the American Mathematical Society. In this paper, I prove that a general subclass of convex measures are $\frac{1}{n}$ -concave under additional assumptions on the measure or on the sets, such as symmetries. This generalizes recent results of Gardner and Zvavitch [41].

4.1 The parallel μ -volume

In this section, we extend the parallel volume in the more general context of s -concave measures, by considering the function $t \mapsto \mu(A + tB_2^n)$, where A is a compact subset of \mathbb{R}^n . We will establish concavity property of this generalized parallel volume, called *parallel μ -volume*. More precisely, given an s -concave measure μ , we will show that in special cases the function $t \mapsto \mu(A + tB_2^n)$ is s -concave on \mathbb{R}_+ , which generalizes the work presented in Chapter 2.

Let $a \in \mathbb{R}^n$. The Dirac measure $\delta_{\{a\}}$ is $+\infty$ -concave and we notice that the function $t \mapsto \delta_{\{a\}}(A + tB_2^n)$ is constant on its support and thus is $+\infty$ -concave on \mathbb{R}_+ . Since the function $t \mapsto \mu(A + tB_2^n)$ is non-decreasing, it follows that the parallel μ -volume is $-\infty$ -concave on \mathbb{R}_+ for every non-negative measure.

Notice that for every s -concave μ , the parallel μ -volume of every convex set $A \subset \mathbb{R}^n$ is s -concave on \mathbb{R}_+ . Indeed, let μ be an s -concave measure and A be a compact convex subset of \mathbb{R}^n , then for every $\lambda \in [0, 1]$ and every $t_1, t_2 \in \mathbb{R}_+$, one has

$$\begin{aligned} \mu(A + ((1 - \lambda)t_1 + \lambda t_2)B_2^n) &= \mu((1 - \lambda)(A + t_1 B_2^n) + \lambda(A + t_2 B_2^n)) \\ &\geq ((1 - \lambda)\mu(A + t_1 B_2^n)^s + \lambda\mu(A + t_2 B_2^n)^s)^{\frac{1}{s}}. \end{aligned}$$

Notice also that the s -concavity of the parallel μ -volume of a fixed compact set A , with μ s -concave, gives a refinement since it then follows that

$$\begin{aligned} \mu((1 - \lambda)(A + t_1 B_2^n) + \lambda(A + t_2 B_2^n)) &\geq \mu(A + ((1 - \lambda)t_1 + \lambda t_2)B_2^n) \\ &\geq ((1 - \lambda)\mu(A + t_1 B_2^n)^s + \lambda\mu(A + t_2 B_2^n)^s)^{\frac{1}{s}}. \end{aligned}$$

Let us study the concavity of the parallel μ -volume in dimension 1. We first establish a preliminary lemma in dimension 1.

Lemma 4.1.1. *Let μ be an s -concave measure on \mathbb{R} which is absolutely continuous with respect to the Lebesgue measure and let A be a compact subset of \mathbb{R} . Then the function $t \mapsto V_A^\mu(t) = \mu(A + t[-1, 1])$ admits left and right derivatives on $(0, +\infty)$, denoted respectively by $(V_A^\mu)'_-$ and by $(V_A^\mu)'_+$, which satisfy for every $t > 0$, $(V_A^\mu)'_-(t) \geq (V_A^\mu)'_+(t)$.*

Proof. Let us denote by ψ the density of the measure μ and let us denote by $[\alpha, \beta]$ the support of μ , with $-\infty \leq \alpha < \beta \leq +\infty$. Notice that for every $t > 0$, the set $A + t[-1, 1]$ is a finite disjoint union of intervals, thus one can assume that $A = \cup_{i=1}^N [a_i, b_i]$, with $\alpha \leq a_1 < b_1 < \dots < a_N < b_N \leq \beta$, and $a_1 = \alpha$ when $\alpha \in A$, $b_N = \beta$ when $\beta \in A$. Let us denote $t_i = \frac{a_{i+1} - b_i}{2}$, $i \in \{0, \dots, N\}$, with the convention that $b_0 = 2\alpha - a_1$ and that $a_{N+1} = 2\beta - b_N$. Notice that V_A^μ is differentiable on $(0, +\infty) \setminus \{t_0, \dots, t_N\}$ and that for every $t > 0$, one has

$$(V_A^\mu)'_+(t) = \sum_{a \in \partial(A+t[-1,1])} \psi(a)1_{(\alpha, \beta)}(a), \quad (V_A^\mu)'_-(t) = \sum_{a \in \partial(A+t(-1,1))} \psi(a)1_{[\alpha, \beta]}(a).$$

Notice that $\overline{A + t[-1, 1]} = \overline{A + t(-1, 1)}$ thus $\partial(A + t[-1, 1]) \subset \partial(A + t(-1, 1))$. We conclude that for every $t > 0$, $(V_A^\mu)'_-(t) \geq (V_A^\mu)'_+(t)$. \square

Theorem 4.1.2. *Let $s \in [-\infty, \frac{1}{2}] \cup [1, +\infty]$. Let μ be an s -concave measure on \mathbb{R} and let A be a compact subset of \mathbb{R} . Then the function $t \mapsto \mu(A + t[-1, 1])$ is s -concave on \mathbb{R}_+ .*

The proof of Theorem 4.1.2 uses the following localization theorem.

Theorem 4.1.3 (Fradelizi, Guédon [35] (see also [36])). *Let n be a positive integer, let K be a compact convex set in \mathbb{R}^n and denote by $\mathcal{P}(K)$ the set of probability measures on \mathbb{R}^n , with support contained in K . Let $f : K \rightarrow \mathbb{R}$ be an upper semi-continuous function, let $s \in [-\infty, \frac{1}{2}]$ and denote by P_f the set of s -concave probability measures μ , with support contained in K , satisfying $\int f d\mu \geq 0$. Let $\Phi : \mathcal{P}(K) \rightarrow \mathbb{R}$ be a convex w^* -upper semi-continuous function. Then $\sup\{\Phi(\mu); \mu \in P_f\}$ is achieved at a probability measure ν which is either a Dirac measure at a point x such that $f(x) \geq 0$, or a probability measure ν which is s -affine on a segment $[\alpha, \beta] \subset K$, such that $\int f d\nu = 0$ and $\int_{\alpha}^x f d\nu > 0$ on (α, β) or $\int_x^{\beta} f d\nu > 0$ on (α, β) .*

Proof of Theorem 4.1.2. Let μ be an s -concave measure on \mathbb{R} and let A be a compact subset of \mathbb{R} .

If $s > 1$ then μ is a Dirac measure δ_x , $x \in \mathbb{R}$, and one can see that the function $t \mapsto \delta_x(A + t[-1, 1])$ is constant on its support. Thus this function is $+\infty$ -concave on \mathbb{R}_+ , which proves Theorem 4.1.2 for $s > 1$. Notice that this argument is also valid in higher dimension.

If $s = 1$ then for every $x \in \mathbb{R}$, $d\mu(x) = C1_{[\alpha, \beta]}(x)dx$, with $C > 0$ is a constant and $[\alpha, \beta]$ is an interval of \mathbb{R} . One can see by a direct computation that $t \mapsto \mu(A + t[-1, 1])$ is 1-concave on \mathbb{R}_+ .

If $s \leq \frac{1}{2}$ then μ admits a density with respect to the Lebesgue measure on \mathbb{R} which is $\frac{s}{1-s}$ -concave. We assume that $s \neq 0$, the case $s = 0$ follows by continuity. Let $t_0 > 0$ such that the set $A + t_0[-1, 1]$ is an interval. Notice that the function $t \mapsto V_A^\mu(t) = \mu(A + t[-1, 1])$ is s -concave on $[t_0, +\infty)$.

To prove that V_A^μ is s -concave on $(0, t_0)$, we shall reduce the problem to extremal measures ν described in Theorem 4.1.3. For these specific measures, the outer parallel volume is easy to compute and is twice differentiable except for a finite number of points. Then we prove by differentiation that the function $t \mapsto V_A^\nu(t) = \nu(A + t[-1, 1])$ is s -concave outside points of non-differentiability. To conclude global s -concavity, we use Lemma 4.1.1.

Step 1: Reduction to extremal measures

Let $t_1, t_2 \in (0, t_0)$ such that $\mu(A + t_1[-1, 1])\mu(A + t_2[-1, 1]) > 0$. We assume that $t_1 < t_2$ without loss of generality. We denote $K = A + t_0[-1, 1]$ and we consider the restriction of μ over K , thus it is a finite measure that we can assume to be a probability measure

without loss of generality. For convenience, we still denote by μ this measure. Let $0 < \varepsilon < t_2 - t_1$. We apply Theorem 4.1.3 to $f : K \rightarrow \mathbb{R}$ defined by

$$f = 1_{A+t_2[-1,1]} - \tau_\varepsilon 1_{A+(t_1+\varepsilon)(-1,1)}$$

and $\Phi_\varepsilon : \mathcal{P}(K) \rightarrow \mathbb{R}$ defined by

$$\Phi_\varepsilon = \rho_\varepsilon 1_{A+t_1[-1,1]} - 1_{A+\frac{(t_1+\varepsilon)+t_2}{2}(-1,1)}$$

where

$$\rho_\varepsilon = \left(\frac{1}{2} \left(\frac{\mu(A+t_2[-1,1])^s}{\mu(A+(t_1+\varepsilon)[-1,1])^s} + 1 \right) \right)^{\frac{1}{s}}, \quad \tau_\varepsilon = \frac{\mu(A+t_2[-1,1])}{\mu(A+(t_1+\varepsilon)[-1,1])}.$$

The choice of introducing ε and the open interval $(-1, 1)$ is a technical trick to obtain upper semi-continuous functions regarding f and Φ_ε and to make the argument work for Dirac measures in Step 2 below. We shall prove that $\Phi_\varepsilon(\mu) \leq 0$, thus by letting ε go to 0 and by using that V_A^μ is continuous on $(0, t_0)$, this will lead to the conclusion that V_A^μ is s -concave on $[t_1, t_2]$, for arbitrary $t_1, t_2 \in (0, t_0)$. To prove that $\Phi_\varepsilon(\mu) \leq 0$, we shall prove that $\Phi_\varepsilon(\nu) \leq 0$ for the extremal probability measures ν described in Theorem 4.1.3. First, notice that $\tau_\varepsilon \geq 1$. If $\tau_\varepsilon = 1$, then V_A^μ is constant on $[t_1 + \varepsilon, t_2]$. Thus V_A^μ is s -concave on $[t_1 + \varepsilon, t_2]$. Thereafter, we assume that $\tau_\varepsilon > 1$.

Step 2: s -concavity for extremal measures

- Let ν be a Dirac measure δ_x with x such that $f(x) \geq 0$. The condition $f(x) \geq 0$ says that $1_{A+t_2[-1,1]}(x) \geq \tau_\varepsilon 1_{A+(t_1+\varepsilon)(-1,1)}(x)$. Since $\tau_\varepsilon > 1$, it follows that $x \notin A + (t_1 + \varepsilon)(-1, 1)$. Thus $x \notin A + t_1[-1, 1]$. Hence $\Phi_\varepsilon(\delta_x) \leq 0$.

- Let ν be an s -affine measure with support $[\alpha, \beta]$, *i.e.* the density of the measure ν , denoted by ψ , satisfies for every $x \in \mathbb{R}$, $\psi(x) = (mx + p)^{1/\gamma} 1_{[\alpha, \beta]}(x)$, where $\gamma = s/(1-s) \in [-1, 1] \setminus \{0\}$ and where $m \in \mathbb{R} \setminus \{0\}$, $p \in \mathbb{R}$ are such that for every $x \in [\alpha, \beta]$, $mx + p \geq 0$. We assume that $\int_{[x, \beta]} f d\nu < 0$ on (α, β) , which means that for every $x \in (\alpha, \beta)$,

$$\nu((A+t_2[-1,1]) \cap [x, \beta]) < \tau_\varepsilon \nu((A+(t_1+\varepsilon)[-1,1]) \cap [x, \beta]). \quad (4.1)$$

If $\beta \notin A + (t_1 + \varepsilon)[-1, 1]$, then there exists $x \in (\alpha, \beta)$ such that $(A + (t_1 + \varepsilon)[-1, 1]) \cap$

$[x, \beta] = \emptyset$. This contradicts (4.1). It follows that $\beta \in A + (t_1 + \varepsilon)[-1, 1]$.

Notice that the function V_A^ν is twice differentiable on $[t_1 + \varepsilon, t_2]$ outside a finite number of points s_0, \dots, s_ℓ , with $s_0 := t_1 + \varepsilon < \dots < s_\ell < s_{\ell+1} := t_2$.

Case 1: $\alpha \in A + (t_1 + \varepsilon)[-1, 1]$.

Let $j \in \{0, \dots, \ell\}$. Notice that $A + s_j[-1, 1]$ is a finite disjoint union of intervals containing α and β , hence we can assume that $A + s_j[-1, 1] = \cup_{i=1}^N [a_i, b_i]$, with $a_1 \leq \alpha < b_1 < \dots < a_N < \beta \leq b_N$. We assume that $N \geq 2$, otherwise the result clearly holds.

Let $t \in [s_j, s_{j+1})$. We have

$$\begin{aligned} V_A^\nu(t) &= \sum_{i=1}^N \int_{a_i-t}^{b_i+t} (mx+p)^{\frac{1}{\gamma}} 1_{[\alpha, \beta]}(x) dx, \\ (V_A^\nu)'(t) &= \sum_{i=2}^N \left((m(b_{i-1}+t)+p)^{\frac{1}{\gamma}} + (m(a_i-t)+p)^{\frac{1}{\gamma}} \right), \\ (V_A^\nu)''(t) &= \frac{m}{\gamma} \left(\sum_{i=2}^N \left((m(b_{i-1}+t)+p)^{\frac{1-\gamma}{\gamma}} - (m(a_i-t)+p)^{\frac{1-\gamma}{\gamma}} \right) \right) \leq 0. \end{aligned}$$

Hence the function V_A^ν is concave on $[s_j, s_{j+1})$. We deduce that V_A^ν is piecewise s -concave on $[t_1 + \varepsilon, t_2]$. From Lemma 4.1.1, we have that for every $t \in [t_1 + \varepsilon, t_2]$,

$$s((V_A^\nu)^s)'_-(t) = s^2(V_A^\nu)'_-(t)(V_A^\nu)^{s-1}(t) \geq s^2(V_A^\nu)'_+(t)(V_A^\nu)^{s-1}(t) = s((V_A^\nu)^s)'_+(t).$$

We conclude that the function V_A^ν is s -concave on $[t_1 + \varepsilon, t_2]$.

Case 2: $\alpha \notin A + (t_1 + \varepsilon)[-1, 1]$.

Let $j \in \{0, \dots, \ell\}$. If $\alpha \in A + s_j[-1, 1]$ then from the previous case we can conclude that V_A^ν is s -concave on $[s_j, s_{j+1})$. Thus we can assume that $A + s_j[-1, 1] = \cup_{i=1}^N [a_i, b_i]$, with $\alpha < a_1 < b_1 < \dots < a_N < \beta \leq b_N$ and that $\alpha \notin A + s_{j+1}[-1, 1]$. We assume that $N \geq 2$, otherwise the result clearly holds. In the following, we denote $a_i(t) = m(a_i - t) + p$, $b_i(t) = m(b_i + t) + p$, $1 \leq i \leq N - 1$ and $a_N(t) = m(a_N - t) + p$, $b_N(t) = m\beta + p$.

Let $t \in [s_j, s_{j+1})$. We have

$$\begin{aligned} V_A^\nu(t) &= \frac{1}{m} \frac{\gamma}{1+\gamma} \sum_{i=1}^N \left(b_i(t)^{\frac{1+\gamma}{\gamma}} - a_i(t)^{\frac{1+\gamma}{\gamma}} \right), \\ (V_A^\nu)'(t) &= \sum_{i=1}^{N-1} \left(b_i(t)^{\frac{1}{\gamma}} + a_i(t)^{\frac{1}{\gamma}} \right) + a_N(t)^{\frac{1}{\gamma}}, \\ (V_A^\nu)''(t) &= \frac{m}{\gamma} \left(-a_1(t)^{\frac{1-\gamma}{\gamma}} + \sum_{i=2}^N \left(b_{i-1}(t)^{\frac{1-\gamma}{\gamma}} - a_i(t)^{\frac{1-\gamma}{\gamma}} \right) \right). \end{aligned}$$

Then the function V_A^ν is s -concave on $[s_j, s_{j+1})$ if and only if for every $t \in [s_j, s_{j+1})$, $V_A^\nu(t)(V_A^\nu)''(t) \leq (1-s)(V_A^\nu)'(t)^2$ if and only if

$$\begin{aligned} \left(\sum_{i=1}^N \left(a_i(t)^{\frac{1+\gamma}{\gamma}} - b_i(t)^{\frac{1+\gamma}{\gamma}} \right) \right) \left(\sum_{i=1}^{N-1} \left(a_i(t)^{\frac{1-\gamma}{\gamma}} - b_i(t)^{\frac{1-\gamma}{\gamma}} \right) + a_N(t)^{\frac{1-\gamma}{\gamma}} \right) \\ \leq \left(\sum_{i=1}^{N-1} \left(b_i(t)^{\frac{1}{\gamma}} + a_i(t)^{\frac{1}{\gamma}} \right) + a_N(t)^{\frac{1}{\gamma}} \right)^2. \end{aligned}$$

If $m\gamma > 0$, then one has

$$0 \leq a_1(t)^{\frac{1-\gamma}{\gamma}} < b_1(t)^{\frac{1-\gamma}{\gamma}} < \dots < a_N(t)^{\frac{1-\gamma}{\gamma}} < b_N(t)^{\frac{1-\gamma}{\gamma}}.$$

Thus $(V_A^\nu)''(t) \leq 0$. Hence the function V_A^ν is concave on $[s_j, s_{j+1})$. Using the same argument as in Case 1, we conclude that the function V_A^ν is s -concave on $[t_1 + \varepsilon, t_2]$.

If $m\gamma < 0$, then one has

$$a_1(t)^{\frac{1-\gamma}{\gamma}} > b_1(t)^{\frac{1-\gamma}{\gamma}} > \dots > a_N(t)^{\frac{1-\gamma}{\gamma}} > b_N(t)^{\frac{1-\gamma}{\gamma}} \geq 0.$$

We deduce that

$$\begin{aligned} \left(\sum_{i=1}^N \left(a_i(t)^{\frac{1+\gamma}{\gamma}} - b_i(t)^{\frac{1+\gamma}{\gamma}} \right) \right) \left(\sum_{i=1}^{N-1} \left(a_i(t)^{\frac{1-\gamma}{\gamma}} - b_i(t)^{\frac{1-\gamma}{\gamma}} \right) + a_N(t)^{\frac{1-\gamma}{\gamma}} \right) \\ \leq a_1(t)^{\frac{1+\gamma}{\gamma}} a_1(t)^{\frac{1-\gamma}{\gamma}} \leq \left(\sum_{i=1}^{N-1} \left(b_i(t)^{\frac{1}{\gamma}} + a_i(t)^{\frac{1}{\gamma}} \right) + a_N(t)^{\frac{1}{\gamma}} \right)^2. \end{aligned}$$

Hence the function V_A^ν is s -concave on $[s_j, s_{j+1})$. We conclude that V_A^ν is s -concave on $[t_1 + \varepsilon, t_2]$.

Hence we deduce that $\Phi_\varepsilon(\nu) \leq 0$ and it follows that V_A^μ is s -concave on $(0, t_0)$. We have seen that V_A^μ is s -concave on $[t_0, +\infty)$ and using Lemma 4.1.1 we conclude that V_A^μ is s -concave on $(0, +\infty)$. Finally, from the non-decreasing property of V_A^μ , it follows that V_A^μ is s -concave on \mathbb{R}_+ . \square

Remark. The result clearly holds if we replace the interval $[-1, 1]$ by any symmetric interval. However it is not necessarily true for an arbitrary interval. For example, let $0 < s \leq \frac{1}{2}$ and take $B = [0, 1]$, $A = [0, 1] \cup [2, 3]$ and $d\mu(x) = x^{\frac{1}{\gamma}} 1_{[0,3]}(x) dx$, with $\gamma = \frac{s}{1-s}$. Then μ is an s -concave measure. For $t \in [0, \frac{1}{2})$ we have

$$V_A^\mu(t) = \mu(A + tB) = \frac{\gamma}{\gamma+1} \left((1+t)^{\frac{\gamma+1}{\gamma}} + 3^{\frac{\gamma+1}{\gamma}} - 2^{\frac{\gamma+1}{\gamma}} \right).$$

Thus,

$$V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 = \frac{1}{\gamma+1} \left(3^{\frac{\gamma+1}{\gamma}} - 2^{\frac{\gamma+1}{\gamma}} \right) > 0.$$

Hence V_A^μ is not s -concave on \mathbb{R}_+ . For $s = 0$, the same example works. For $s < 0$, one can take $B = [-1, 0]$, $A = [0, 1] \cup [2, 3]$ and $d\mu(x) = x^{\frac{1}{\gamma}} 1_{[\alpha,3]}(x) dx$, with $\gamma = \frac{s}{1-s}$ and α sufficiently small.

The localization theorem (Theorem 4.1.3) holds only for $s \leq \frac{1}{2}$. Thus this theorem could not be used to examine the case of s -concave measures, with $s > \frac{1}{2}$. In fact, for $s \in (\frac{1}{2}, 1)$, the parallel μ -volume is not s -concave in general as shown in Proposition 4.1.4 below, but under specific conditions regarding the support of the measure, we can show that a stronger positive result holds. First, let us show that for $s \in (\frac{1}{2}, 1)$ the parallel μ -volume is not s -concave in dimension 1.

Proposition 4.1.4. *Let $s \in (\frac{1}{2}, 1)$ and let $\gamma = \frac{s}{1-s}$. Let $\beta = 10(1 - 2^{\frac{1-\gamma}{\gamma}})^{-1}$ and let μ be a measure such that $d\mu(x) = x^{\frac{1}{\gamma}} 1_{[0,\beta]}(x) dx$. Let us set $A = [0, 1] \cup [2, \beta]$. Then the function $t \mapsto V_A^\mu(t) = \mu(A + t[-1, 1])$ is not s -concave on \mathbb{R}_+ .*

Proof. For every $t \in [0, \frac{1}{2})$,

$$\begin{aligned} V_A^\mu(t) &= \frac{\gamma}{\gamma+1} \left((1+t)^{\frac{1+\gamma}{\gamma}} + \beta^{\frac{1+\gamma}{\gamma}} - (2-t)^{\frac{1+\gamma}{\gamma}} \right), \\ (V_A^\mu)'(t) &= (1+t)^{\frac{1}{\gamma}} + (2-t)^{\frac{1}{\gamma}}, \\ (V_A^\mu)''(t) &= \frac{1}{\gamma} \left((1+t)^{\frac{1-\gamma}{\gamma}} - (2-t)^{\frac{1-\gamma}{\gamma}} \right). \end{aligned}$$

Hence,

$$V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 = \frac{1}{\gamma+1} \left(\beta^{\frac{1+\gamma}{\gamma}} (1 - 2^{\frac{1-\gamma}{\gamma}}) - 2^{\frac{1-\gamma}{\gamma}} - 2^{\frac{1+2\gamma}{\gamma}} \right).$$

Since $1 - 2^{\frac{1-\gamma}{\gamma}} > 0$ and $\beta > \left((2^{\frac{1+2\gamma}{\gamma}} + 2^{\frac{1-\gamma}{\gamma}}) / (1 - 2^{\frac{1-\gamma}{\gamma}}) \right)^{\frac{\gamma}{\gamma+1}}$, it follows that $V_A^\mu(0)(V_A^\mu)''(0) - (1-s)(V_A^\mu)'(0)^2 > 0$. We conclude that V_A^μ is not s -concave on \mathbb{R}_+ . \square

We denote by $\text{supp}(\mu)$ the support of μ and by $\text{dist}(A, \text{supp}(\mu)^c)$ the distance between A and the complement of the support of μ . When the support of μ is \mathbb{R} , the distance will be equal to $+\infty$.

Proposition 4.1.5. *Let $s \geq \frac{1}{2}$. Let μ be an s -concave measure on \mathbb{R} which is absolutely continuous with respect to the Lebesgue measure. Let A be a compact subset of \mathbb{R} such that $\text{dist}(A, \text{supp}(\mu)^c) > 0$. Then the function $t \mapsto V_A^\mu(t) = \mu(A + t[-1, 1])$ is concave on $[0, \text{dist}(A, \text{supp}(\mu)^c))$.*

Proof. First, we assume that $s = \frac{1}{2}$. Hence μ admits a 1-concave density function denoted by ψ . Notice that V_A^μ is differentiable outside a finite number of points t_0, \dots, t_N and that for every $t \in [0, \text{dist}(A, \text{supp}(\mu)^c)) \setminus \{t_0, \dots, t_N\}$ one has

$$V_A^\mu(t) = \sum_{i=1}^N \int_{a_i-t}^{b_i+t} \psi(x) dx, \quad (V_A^\mu)'(t) = \sum_{i=1}^N (\psi(b_i+t) + \psi(a_i-t)).$$

Since ψ is concave, it follows that for every $i \in \{1, \dots, N\}$, the function $t \mapsto \psi(b_i+t) + \psi(a_i-t)$ is non-increasing. Thus $(V_A^\mu)'$ is piecewise non-increasing. We conclude that V_A^μ is piecewise concave on $[0, \text{dist}(A, \text{supp}(\mu)^c))$. From Lemma 4.1.1, we deduce that V_A^μ is concave on $[0, \text{dist}(A, \text{supp}(\mu)^c))$.

Finally, if μ is s -concave with $s \geq \frac{1}{2}$, then μ is $\frac{1}{2}$ -concave and we conclude from the first part of the proof that V_A^μ is concave on $[0, \text{dist}(A, \text{supp}(\mu)^c))$. \square

Now we study the parallel μ -volume in dimension $n \geq 2$. We have seen in Proposition 2.3.3 that the Costa-Cover conjecture 2.1.1 is false in dimension $n \geq 2$. Hence, even for the classical parallel volume, one cannot expect such concavity in general. Let us recall the counterexample.

Let $n \geq 2$. Let us set $A = B_2^n \cup \{2e_1\}$ and let us denote $V_A(t) = |A + tB_2^n|$. Then for every $t \in [0, \frac{1}{2})$, one has

$$V_A(t) = |B_2^n \cup \{2e_1\} + tB_2^n| = |B_2^n + tB_2^n| + |tB_2^n| = |B_2^n|((1+t)^n + t^n).$$

Since the $\frac{1}{n}$ -power of this function is not concave (it is strictly convex), V_A is not $\frac{1}{n}$ -concave on \mathbb{R}_+ for $n \geq 2$.

It could appear surprising that the parallel μ -volume is not s -concave in dimension $n \geq 2$ since we proved that this is the case in dimension 1 with the localization theorem (when $s \leq \frac{1}{2}$). This localization technique is usually used to reduce inequalities for general convex measures in dimension n to inequalities for measures for which the support is a segment, thus the problem becomes 1-dimensional (see *e.g.* [44], [33]). Let us explain why one cannot reduce our problem to the dimension 1. The reduction done in dimension 1 with localization works the same way in dimension n and we have the following equivalence for every compact set A of \mathbb{R}^n :

- i) V_A^μ is s -concave for every measure μ s -concave.
- ii) V_A^ν is s -concave for every measure ν s -affine on a segment $[\alpha, \beta]$.

However, ii) is not true in dimension $n \geq 2$. Indeed, we can construct an explicit counterexample to show that the function $t \mapsto |(A + tB_2^n) \cap [\alpha, \beta]|_1$ is not continuous everywhere inside its support and hence this function is not s -concave. For example, consider $A = \{(0, 0)\} \cup \{(3, 0)\} \cup \{(x, 1); x \in [1, 2]\}$ and $[\alpha, \beta] = \{(x, 0); x \in [0, 3]\}$.

For the classical parallel volume, we have seen in dimension 2 that if A is a connected subset of \mathbb{R}^2 then the function $t \mapsto |A + tB_2^2|$ is $\frac{1}{2}$ -concave on \mathbb{R}_+ (see Theorem 2.3.2). However, the following proposition shows that this property fails in general if we replace the Lebesgue measure by an arbitrary s -concave measure.

Proposition 4.1.6. *In dimension 2, there exists a connected set A and a $\frac{1}{2}$ -concave measure μ such that $t \mapsto \mu(A + tB_2^2)$ is not $\frac{1}{2}$ -concave on \mathbb{R}_+ .*

Proof. We set $d\mu(x) = 1_{B_1^2}(x)dx$, where B_1^2 denotes the unit ball for the ℓ_1^2 norm. Hence μ is $\frac{1}{2}$ -concave. We construct the points $B = (-1, 0)$, $C = (-0.5, -0.5)$, $D = (0.5, 0.5)$, $E = (0, 1)$, $F = (-2, 0)$, $G = (0, -2)$, $H = (0, -1)$, $I = (2, 0)$, $J = (1, 0)$. We set

$$A = \text{conv}(BCDE) \cup [FB] \cup [FG] \cup [GH] \cup [GI] \cup [IJ].$$



Figure 4.1: Counterexample

Then A is connected and for every $t \in [0, \frac{1}{8}]$, we have

$$V_A^\mu(t) = \mu(A + tB_2^2) = \frac{\sqrt{2}}{2} + \sqrt{2}t + \frac{\pi}{2}t^2.$$

It follows that $\left(\sqrt{V_A^\mu}\right)''(0) > 0$. We conclude that $t \mapsto \mu(A + tB_2^2)$ is not $\frac{1}{2}$ -concave on \mathbb{R}_+ . \square

Remark. Notice that we can adapt the counterexample of Proposition 4.1.6 to show that there exists an s -concave measure μ on \mathbb{R}^n , $n \geq 2$, such that for every $r \in (-\infty, s)$ there exists a compact connected set $A \subset \mathbb{R}^n$ such that $t \mapsto \mu(A + tB_2^n)$ is not r -concave on \mathbb{R}_+ .

4.2 On the improvement of concavity of convex measures

Recently, Gardner and Zvavitch [41] proved that the Gaussian measure γ_n on \mathbb{R}^n , defined by

$$d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{|x|^2}{2}} dx, \quad x \in \mathbb{R}^n$$

where $|\cdot|$ denotes the Euclidean norm, satisfies the Brunn-Minkowski inequality for some special classes of sets. More precisely, Gardner and Zvavitch showed that for coordinate boxes (*i.e.* a product of intervals) $A, B \subset \mathbb{R}^n$ that contain the origin, or for $A, B \subset \mathbb{R}^n$ that are dilates of the same symmetric convex set, and for every $\lambda \in [0, 1]$, one has

$$\gamma_n((1 - \lambda)A + \lambda B)^{\frac{1}{n}} \geq (1 - \lambda)\gamma_n(A)^{\frac{1}{n}} + \lambda\gamma_n(B)^{\frac{1}{n}}. \quad (4.2)$$

They also conjectured that inequality (4.2) holds for all convex symmetric sets $A, B \subset \mathbb{R}^n$.

As a consequence of the Prékopa-Leindler inequality (1.8) [70], [60], [71], the Gaussian measure satisfies for all Borel sets $A, B \subset \mathbb{R}^n$ and for every $\lambda \in [0, 1]$,

$$\gamma_n((1 - \lambda)A + \lambda B) \geq \gamma_n(A)^{1-\lambda}\gamma_n(B)^\lambda. \quad (4.3)$$

Using the terminology of Borell [15] (see Chapter 1 for further details), inequality (4.3) means that the Gaussian measure is a log-concave measure. By comparing means, one sees that inequality (4.2) is stronger than inequality (4.3), hence the results of Gardner and Zvavitch improve the concavity of the Gaussian measure by showing that this measure is $\frac{1}{n}$ -concave if restricted to special classes of sets.

We will see that the results of Gardner and Zvavitch can be extended to the more general class of convex measures. This is the mathematical underlying idea of the Gaussian Brunn-Minkowski inequality (4.2), *i.e.* under symmetry assumptions one can improve a certain property; in this case it is the concavity of a measure. However, we will see that symmetries are not the only geometric hypothesis that improve concavity properties of a measure.

This section is devoted to the study of the following question:

Question 1. For which value $s \in [-\infty, +\infty]$, for which class \mathcal{M} of measures on \mathbb{R}^n and for which class \mathcal{C} of couples of Borel subsets of \mathbb{R}^n one has

$$\mu((1-\lambda)A + \lambda B) \geq ((1-\lambda)\mu(A)^s + \lambda\mu(B)^s)^{\frac{1}{s}} \quad (4.4)$$

for every $\mu \in \mathcal{M}$, for every $(A, B) \in \mathcal{C}$ such that $\mu(A)\mu(B) > 0$ and for every $\lambda \in [0, 1]$?

Borell [15] (see also [18]) answered Question 1 when \mathcal{M} is the class of s -concave measures on \mathbb{R}^n , $s \in [-\infty, +\infty]$, and when \mathcal{C} is the class of all couples of Borel subsets of \mathbb{R}^n (see Chapter 1).

Question 1 has been explored for $s = 1$, when restricted to the Lebesgue measure, by Bonnesen [13] and is still being studied (see *e.g.* [49]).

In the following, we study Question 1.

Preliminaries

A subset $A \subset \mathbb{R}^n$ is said to be *symmetric* if $A = -A$. A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be *unconditional* if for every $(x_1, \dots, x_n) \in \mathbb{R}^n$ and for every $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$, one has $f(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) = f(x_1, \dots, x_n)$. A subset $A \subset \mathbb{R}^n$ is said to be *unconditional* if the indicator function of A , denoted by 1_A , is unconditional. A measure with density function is said to be *symmetric* (resp. *unconditional*) if its density function is even (resp. unconditional).

Measures with $-\infty$ -concave density function are a natural generalization of convex measures. The results of Borell (Theorem 1.7.1) show that a measure that has a γ -concave density function with $\gamma < -\frac{1}{n}$, does not satisfy a concavity property of the form (1.23) (but satisfies other forms of concavity [26]). However, we will show that if restricted to a special class of sets, such measures are $\frac{1}{n}$ -concave.

We call *sub-convex measure* a measure with $-\infty$ -concave density. Notice that convex measures are sub-convex.

Brunn-Minkowski-type inequality for sub-convex measures

In this section, we partially answer Question 1 by investigating possible improvements of the concavity of sub-convex measures. Gardner and Zvavitch [41] noticed in the case

of the Gaussian measure, that the position of the sets A and B plays an important role. Indeed, since for s -concave probability measures μ , with $s \leq 0$, the density function tends to 0 at infinity and the support can be equal to \mathbb{R}^n , one can find sets A and B such that A contains the origin and $\frac{A+B}{2}$ is far from the origin. Thus for $r > 0$, the inequality

$$\mu\left(\frac{A+B}{2}\right)^r \geq \frac{\mu(A)^r + \mu(B)^r}{2}$$

will not be satisfied. Hence, the position of sets A and B is an inherent constraint of the problem. Notice also that in the definition of s -concave measures, the condition $\mu(A)\mu(B) > 0$ is already a constraint on the position of A and B with respect to the support of μ .

Notice that Question 1 has an answer for $s = +\infty$ if \mathcal{M} is the class of convex measures and if \mathcal{C} is the class of couples of Borel sets with the same measure. Indeed, one then has for every $\lambda \in [0, 1]$,

$$\mu((1-\lambda)A + \lambda B) \geq \inf(\mu(A), \mu(B)),$$

by definition. Since $\mu(A) = \mu(B)$, it follows that

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A) = \max(\mu(A), \mu(A)) = \max(\mu(A), \mu(B)).$$

Notice also that for every measure μ and for all Borel sets A, B such that $A \subset B$, one has for every $\lambda \in [0, 1]$,

$$\mu((1-\lambda)A + \lambda B) \geq \min(\mu(A), \mu(B)),$$

since in this case one has, $(1-\lambda)A + \lambda B \supset (1-\lambda)A + \lambda A \supset A$.

The case of symmetric measures and symmetric sets

Under symmetry assumptions, the best concavity one can obtain is $\frac{1}{n}$ by considering, for example, the Lebesgue measure, which fulfills a lot of symmetries (unconditional), and two dilates of B_2^n (which are unconditional). This was noticed by Gardner and Zvavitch [41] also for the Gaussian measure.

A sufficient condition to ensure that a measure μ on \mathbb{R}^n is $\frac{1}{n}$ -concave in the class of dilates of a fixed Borel set $A \subset \mathbb{R}^n$ is that the function $t \mapsto \mu(tA)$ is $\frac{1}{n}$ -concave. The following proposition gives a sufficient condition for this.

Proposition 4.2.1. *Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a measurable function such that for every $x \in \mathbb{R}^n$, the function $t \mapsto \phi(tx)$ is non-increasing on \mathbb{R}_+ . Let μ be a measure with density function ϕ and let A be a Borel subset of \mathbb{R}^n containing 0. If the function $t \mapsto \mu(e^t A)$ is log-concave on \mathbb{R} , then the function $t \mapsto \mu(tA)$ is $\frac{1}{n}$ -concave on \mathbb{R}_+ .*

Proof. Let μ be a measure with density function ϕ satisfying the assumptions of Proposition 4.2.1 and let A be a Borel subset of \mathbb{R}^n containing 0. Let us denote $F(t) = \mu(tA)$, for $t \in \mathbb{R}_+$. Notice that F is non-decreasing and continuous on \mathbb{R}_+ . By assumption, the function $t \mapsto F(e^t)$ is log-concave on \mathbb{R} . It follows that the right derivative of F , denoted by F'_+ , exists everywhere and that $t \mapsto tF'_+(t)/F(t)$ is non-increasing on $(0, +\infty)$.

Notice that the function F is $\frac{1}{n}$ -concave on \mathbb{R}_+ if and only if the function

$$t \mapsto \frac{tF'_+(t)}{F(t)} \frac{F(t)^{\frac{1}{n}}}{t}$$

is non-increasing on \mathbb{R}_+ . A direct change of variables shows that

$$\frac{F(t)}{t^n} = \int_A \phi(tx) dx.$$

By assumption, the function $t \mapsto \phi(tx)$ is non-increasing on \mathbb{R}_+ . It follows that the function $t \mapsto F(t)^{\frac{1}{n}}/t$ is non-increasing on $(0, +\infty)$. Hence, the function $t \mapsto (F(t)^{\frac{1}{n}})'_+$ is non-increasing on $(0, +\infty)$ as the product of two non-negative non-increasing functions on $(0, +\infty)$. We conclude that F is $\frac{1}{n}$ -concave on \mathbb{R}_+ . \square

Remarks.

1. Proposition 4.2.1 is established in [41] for the Gaussian measure by differentiating twice.
2. The assumption $t \mapsto \phi(tx)$ is non-increasing on \mathbb{R}_+ is satisfied if ϕ is an even $-\infty$ -concave function.
3. The converse of Proposition 2.2.2 is false in general, by taking, for example, $\phi(x) = 1_{[-1,2]}(x)$, $x \in \mathbb{R}$, and $A = [-1, 1]$.

Proposition 4.2.1 is related to the (B)-conjecture. This conjecture was posed by W. Banaszczyk [58] and asks whether the function $t \mapsto \gamma_n(e^t A)$ is log-concave on \mathbb{R} , for every convex symmetric set $A \subset \mathbb{R}^n$. The (B)-conjecture was proved by Cordero-Erausquin, Fradelizi and Maurey in [21]. In the same paper [21], the authors have also shown that for every unconditional log-concave measure μ on \mathbb{R}^n and for every unconditional convex subset $A \subset \mathbb{R}^n$, the function $t \mapsto \mu(e^t A)$ is log-concave on \mathbb{R} . Using this and the point 2 of the previous remark, we may apply Proposition 4.2.1 to obtain the following corollary:

Corollary 4.2.2. *Let μ be an unconditional log-concave measure on \mathbb{R}^n and let A be an unconditional convex subset of \mathbb{R}^n . Then, the measure μ is $\frac{1}{n}$ -concave in the class of dilates of A . More precisely, for every $A_1, A_2 \in \{\alpha A; \alpha > 0\}$ and for every $\lambda \in [0, 1]$, we obtain*

$$\mu((1 - \lambda)A_1 + \lambda A_2)^{\frac{1}{n}} \geq (1 - \lambda)\mu(A_1)^{\frac{1}{n}} + \lambda\mu(A_2)^{\frac{1}{n}}.$$

Remark. Very recently, Livne Bar-on [61] and Saroglou [74] proved, using different methods, that in dimension 2 for the uniform measure μ_K on a symmetric convex set $K \subset \mathbb{R}^2$ (i.e. $d\mu_K(x) = 1_K(x) dx$), the function $t \mapsto \mu_K(e^t A)$ is log-concave on \mathbb{R} for every symmetric convex set $A \subset \mathbb{R}^2$. However, for our problem, this information is not useful since the uniform measure on a convex subset of \mathbb{R}^n is a $\frac{1}{n}$ -concave measure.

A natural question is to ask if the role of the symmetry can be relaxed. When restricted to the Gaussian measure, it has been shown by Nayar and Tkocz in [67], that in dimension 2 there exist non-symmetric convex sets $A, B \subset \mathbb{R}^2$ satisfying $0 \in A \subset B$ and

$$\gamma_2\left(\frac{A+B}{2}\right)^{\frac{1}{2}} < \frac{\gamma_2(A)^{\frac{1}{2}} + \gamma_2(B)^{\frac{1}{2}}}{2}. \tag{4.5}$$

One can then construct an explicit counterexample in every dimension $n \geq 2$. Moreover, the counterexample in [67] shows more than inequality (4.5). It shows that

$$\gamma_2\left(\frac{A+B}{2}\right)^s < \frac{\gamma_2(A)^s + \gamma_2(B)^s}{2}, \tag{4.6}$$

for every $s \geq 1 - \frac{2}{\pi}$. However, it could be of interest to know what happens when $s \in (0, 1 - \frac{2}{\pi})$.

Notice that the same counterexample with the following log-concave unconditional measure instead of the Gaussian measure

$$d\mu(x, y) = e^{-|x|}e^{-|y|} dx dy, \quad (x, y) \in \mathbb{R}^2$$

satisfies inequality (4.6) for every $s > 0$.

Thus, in general, the symmetry assumption on the measure is not sufficient.

On the other hand, the concavity of a non-symmetric convex measure cannot be improved in general in the class of symmetric sets even in dimension 1:

Proposition 4.2.3. *Let $0 < s < 1$ and $r > s$. There exists an s -concave measure μ on \mathbb{R} and symmetric sets $A, B \subset \mathbb{R}$ such that*

$$\mu\left(\frac{A+B}{2}\right) < \left(\frac{\mu(A)^r + \mu(B)^r}{2}\right)^{\frac{1}{r}}.$$

Proof. Let us define $d\mu(x) = x^{1/\gamma}1_{\mathbb{R}_+}(x) dx$, with $\gamma = \frac{s}{1-s} > 0$. Let us consider the sets $A = [-a, a]$ and $B = [-b, b]$ with $0 < a < b$. Notice that

$$\lim_{a \rightarrow 0} \mu\left(\frac{A+B}{2}\right) = \mu\left(\frac{B}{2}\right) = \frac{\mu(B)}{2^{\frac{1}{s}}} = \lim_{a \rightarrow 0} \left(\frac{\mu(A)^s + \mu(B)^s}{2}\right)^{\frac{1}{s}}.$$

Since $\mu(A) \neq \mu(B)$, it follows by comparing means that

$$\left(\frac{\mu(A)^s + \mu(B)^s}{2}\right)^{\frac{1}{s}} < \left(\frac{\mu(A)^r + \mu(B)^r}{2}\right)^{\frac{1}{r}}.$$

we conclude that for sufficiently small a ,

$$\mu\left(\frac{A+B}{2}\right) < \left(\frac{\mu(A)^r + \mu(B)^r}{2}\right)^{\frac{1}{r}}.$$

□

Thus, in general, the symmetry assumption on the sets is not sufficient.

The case of sets with a maximal section of equal measure

In this section, we consider \mathcal{C} to be the class of couples of Borel subsets of \mathbb{R}^n having a maximal section of equal measure. A famous result of Bonnesen [13] (initially for convex

sets) states that if $A, B \subset \mathbb{R}^n$ satisfy

$$\sup_{t \in \mathbb{R}} |A \cap (u^\perp + tu)|_{n-1} = \sup_{t \in \mathbb{R}} |B \cap (u^\perp + tu)|_{n-1},$$

for a certain $u \in \mathcal{S}^{n-1}$, where $|\cdot|_{n-1}$ denotes the $(n-1)$ -dimensional Lebesgue measure, then for every $\lambda \in [0, 1]$, one has

$$|(1-\lambda)A + \lambda B| \geq (1-\lambda)|A| + \lambda|B|.$$

There exists a functional version of Bonnesen's result established by Henstock and Macbeath [48] in dimension 1 (see Proposition 4.2.4 below) and later on by Dancs and Uhrin [26] in higher dimension (see Proposition 4.2.8 below).

Proposition 4.2.4 (Henstock, Macbeath [48]). *Let $\lambda \in [0, 1]$. Let $f, g, h : \mathbb{R} \rightarrow \mathbb{R}_+$ be non-negative measurable functions such that $\max(f) = \max(g)$ and such that for every $x, y \in \mathbb{R}$*

$$h((1-\lambda)x + \lambda y) \geq \min(f(x), g(y)).$$

Then, one has

$$\int_{\mathbb{R}} h(x) \, dx \geq (1-\lambda) \int_{\mathbb{R}} f(x) \, dx + \lambda \int_{\mathbb{R}} g(x) \, dx.$$

We deduce the following result:

Proposition 4.2.5. *Let $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$ be a $-\infty$ -concave function such that $\max(\phi)$ is attained at $a \in \mathbb{R}$. Let μ be a measure with density function ϕ . Let A, B be Borel subsets of \mathbb{R} such that $a \in A \cap B$. Then, for every $\lambda \in [0, 1]$, we have*

$$\mu((1-\lambda)A + \lambda B) \geq (1-\lambda)\mu(A) + \lambda\mu(B).$$

Proof. Let $\lambda \in [0, 1]$. We define, for every $x \in \mathbb{R}$, $h(x) = \phi(x)1_{(1-\lambda)A + \lambda B}(x)$, $f(x) = \phi(x)1_A(x)$, $g(x) = \phi(x)1_B(x)$. Notice that for every $x, y \in \mathbb{R}$ one has

$$h((1-\lambda)x + \lambda y) \geq \min(f(x), g(y)),$$

and $\max(f) = \max(g) = \phi(a)$. It follows from Proposition 4.2.4 that

$$\int_{\mathbb{R}} h(x) \, dx \geq (1-\lambda) \int_{\mathbb{R}} f(x) \, dx + \lambda \int_{\mathbb{R}} g(x) \, dx.$$

In other words, we obtain

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

□

Remark. Proposition 4.2.5 was established in [41] for the case where μ is the Gaussian measure on \mathbb{R} and where $A, B \subset \mathbb{R}$ are convex. In the same paper, the authors were able to remove the convexity assumption for only one set, by using long computations. Our method bypasses the use of geometric tools and relies on the functional version Proposition 4.2.4.

Conversely, if a measure μ on \mathbb{R} , with density function ϕ with respect to the Lebesgue measure, satisfies

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B),$$

for every $\lambda \in [0, 1]$ and for all symmetric convex sets $A, B \subset \mathbb{R}$, then one has for every $\lambda \in [0, 1]$ and for every $a, b \in \mathbb{R}_+$,

$$\int_{-((1-\lambda)a+\lambda b)}^{(1-\lambda)a+\lambda b} \phi(x) dx \geq (1 - \lambda) \int_{-a}^a \phi(x) dx + \lambda \int_{-b}^b \phi(x) dx.$$

It follows that the function $t \mapsto \phi(t) + \phi(-t)$ is non-increasing on \mathbb{R}_+ . Notice that this condition is satisfied for more general functions than $-\infty$ -concave functions attaining the maximum at 0.

However, one can use the same argument to see that if one assumes $A, B \subset \mathbb{R}$ convex containing 0 (not necessarily symmetric), then it follows that the density function ϕ is necessarily non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Notice that this is equivalent to the fact that the density function ϕ is $-\infty$ -concave and $\max(\phi)$ is attained at 0.

By tensorization, Proposition 4.2.5 leads to the following corollary:

Corollary 4.2.6. *Let μ_i , $1 \leq i \leq n$, be measures with densities $\phi_i : \mathbb{R} \rightarrow \mathbb{R}_+$ such that ϕ_i are non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Let μ be the product measure of μ_1, \dots, μ_n and let $A, B \subset \mathbb{R}^n$ be the product of n Borel subsets of \mathbb{R} such that*

$0 \in A \cap B$. Then, for every $\lambda \in [0, 1]$, we have

$$\mu((1 - \lambda)A + \lambda B)^{\frac{1}{n}} \geq (1 - \lambda)\mu(A)^{\frac{1}{n}} + \lambda\mu(B)^{\frac{1}{n}}.$$

Proof. We follow [41]. By assumption, $A = \Pi_{i=1}^n A_i$ and $B = \Pi_{i=1}^n B_i$, where for every $i \in \{1, \dots, n\}$, A_i and B_i are Borel subsets of \mathbb{R} containing 0. Let $\lambda \in [0, 1]$. Notice that

$$(1 - \lambda)A + \lambda B = \Pi_{i=1}^n ((1 - \lambda)A_i + \lambda B_i).$$

Using Proposition 4.2.5 and an inequality of Minkowski (see e.g. [46]), one deduces that

$$\begin{aligned} \mu((1 - \lambda)A + \lambda B)^{\frac{1}{n}} &= (\Pi_{i=1}^n \mu_i((1 - \lambda)A_i + \lambda B_i))^{\frac{1}{n}} \\ &\geq (\Pi_{i=1}^n ((1 - \lambda)\mu_i(A_i) + \lambda\mu_i(B_i)))^{\frac{1}{n}} \\ &\geq (\Pi_{i=1}^n (1 - \lambda)\mu_i(A_i))^{\frac{1}{n}} + (\Pi_{i=1}^n \lambda\mu_i(B_i))^{\frac{1}{n}} \\ &= (1 - \lambda)\mu(A)^{\frac{1}{n}} + \lambda\mu(B)^{\frac{1}{n}}. \end{aligned}$$

□

Another consequence of Proposition 4.2.5 is that certain particular product measures are concave measures if A is a union of parallel slabs containing the origin.

Corollary 4.2.7. *Let μ_1 be a measure with density function $\phi : \mathbb{R} \rightarrow \mathbb{R}_+$, such that ϕ is non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Let μ_2 be a $(n - 1)$ -dimensional measure and let μ be the product measure of μ_1 and μ_2 . Let $A = A_1 \times \mathbb{R}^{n-1}$, where A_1 is a Borel subset of \mathbb{R} and let B be a Borel subset of \mathbb{R}^n such that $0 \in A \cap B$. Then, for every $\lambda \in [0, 1]$, we have*

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B). \tag{4.7}$$

Inequality (4.7) was established in [41] with power $1/n$ for slabs, for the case where μ is the Gaussian measure.

Proof. We follow [41]. Let us denote $B_S = P_{e_1}(B) \times \mathbb{R}^{n-1}$, where $P_{e_1}(B)$ denotes the orthogonal projection of B onto the first coordinate axis. Then, for every $\lambda \in [0, 1]$, one has

$$(1 - \lambda)A + \lambda B = (1 - \lambda)A + \lambda B_S.$$

It follows, using Proposition 4.2.5, that

$$\begin{aligned}
\mu((1-\lambda)A + \lambda B) &= \mu((1-\lambda)A + \lambda B_S) \\
&= \mu(((1-\lambda)A_1 + \lambda P_{e_1}(B)) \times \mathbb{R}^{n-1}) \\
&= \mu_1((1-\lambda)A_1 + \lambda P_{e_1}(B)) \cdot \mu_2(\mathbb{R}^{n-1}) \\
&\geq ((1-\lambda)\mu_1(A_1) + \lambda\mu_1(P_{e_1}(B))) \cdot \mu_2(\mathbb{R}^{n-1}) \\
&= (1-\lambda)\mu(A) + \lambda\mu(B_S) \\
&\geq (1-\lambda)\mu(A) + \lambda\mu(B).
\end{aligned}$$

□

On the other hand, Proposition 4.2.4 can be generalized to the dimension n . First, let us define for a non-negative measurable function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ and for $u \in \mathcal{S}^{n-1}$,

$$m_u(f) = \sup_{t \in \mathbb{R}} \int_{u^\perp} f(x + tu) \, dx.$$

Proposition 4.2.8 (Dancs, Uhrin [26]). *Let $-\frac{1}{n-1} \leq \gamma \leq +\infty$, $\lambda \in [0, 1]$ and $f, g, h : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be non-negative measurable functions such that for every $x, y \in \mathbb{R}^n$,*

$$h((1-\lambda)x + \lambda y) \geq ((1-\lambda)f(x)^\gamma + \lambda g(y)^\gamma)^{\frac{1}{\gamma}}.$$

If there exists $u \in \mathcal{S}^{n-1}$ such that $m_u(f) = m_u(g) < +\infty$, then

$$\int_{\mathbb{R}^n} h(x) \, dx \geq (1-\lambda) \int_{\mathbb{R}^n} f(x) \, dx + \lambda \int_{\mathbb{R}^n} g(x) \, dx.$$

Let us denote for a measure μ with density function ϕ , for a Borel subset $A \subset \mathbb{R}^n$ and for a hyperplane $H \subset \mathbb{R}^n$,

$$\mu_{n-1}(A \cap H) = \int_{A \cap H} \phi(x) \, dx.$$

We deduce the following result:

Proposition 4.2.9. *Let μ be a measure with density function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that ϕ is $-\frac{1}{n-1}$ -concave. Let A, B be Borel subsets of \mathbb{R}^n . If there exists $u \in \mathcal{S}^{n-1}$ such that*

$$\sup_{t \in \mathbb{R}} \mu_{n-1}(A \cap (u^\perp + tu)) = \sup_{t \in \mathbb{R}} \mu_{n-1}(B \cap (u^\perp + tu)),$$

then, for every $\lambda \in [0, 1]$, we have

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

Proof. Let $\lambda \in [0, 1]$. Let us take $f = \phi 1_A$, $g = \phi 1_B$ and $h = \phi 1_{(1-\lambda)A + \lambda B}$. Then, for every $x, y \in \mathbb{R}^n$, one has

$$h((1 - \lambda)x + \lambda y) \geq ((1 - \lambda)f(x)^\gamma + \lambda g(y)^\gamma)^{\frac{1}{\gamma}},$$

where $\gamma = -\frac{1}{n-1}$. Moreover,

$$\int_{u^\perp} f(x + tu) \, dx = \int_{A \cap (u^\perp + tu)} \phi(x) \, dx = \mu_{n-1}(A \cap (u^\perp + tu)).$$

It follows that $m_u(f) = m_u(g)$. From Proposition 4.2.8, we obtain that

$$\mu((1 - \lambda)A + \lambda B) \geq (1 - \lambda)\mu(A) + \lambda\mu(B).$$

□

Application to the parallel volume

Let us see how improvements of the concavity of sub-convex measures can improve the concavity of a generalized form of the parallel volume. The parallel volume of a Borel subset A of \mathbb{R}^n , namely the function $t \mapsto |A + tB_2^n|$, is a particularly useful function in Geometry, which has been highlighted by the precursor works of Steiner [81]. For more modern applications, the parallel volume and its generalized forms are still studied (see e.g. [51], [54]). Moreover, this notion of parallel volume leads to the powerful theory of mixed volumes (see [77] for further details).

As a consequence of the Brunn-Minkowski inequality (1.7), one can see that if $A \subset \mathbb{R}^n$ is convex, then the parallel volume of A is $\frac{1}{n}$ -concave on \mathbb{R}_+ . More generally, if a measure μ is s -concave, with $s \in [-\infty; +\infty]$, in the class of sets of the form $\{A + tB; t \in \mathbb{R}_+\}$, where A and B are convex subsets of \mathbb{R}^n , then the generalized parallel volume $t \mapsto \mu(A + tB)$ is s -concave on \mathbb{R}_+ . Indeed, for every $t_1, t_2 \in \mathbb{R}_+$ and for every $\lambda \in [0, 1]$,

one has

$$\begin{aligned} \mu(A + ((1 - \lambda)t_1 + \lambda t_2)B) &= \mu((1 - \lambda)(A + t_1B) + \lambda(A + t_2B)) \\ &\geq ((1 - \lambda)\mu(A + t_1B)^s + \lambda\mu(A + t_2B)^s)^{\frac{1}{s}}. \end{aligned}$$

Using this and Corollary 4.2.6, we obtain the following corollary:

Corollary 4.2.10. *Let μ_i , $1 \leq i \leq n$, be measures with densities $\phi_i : \mathbb{R} \rightarrow \mathbb{R}_+$ such that ϕ_i are non-decreasing on $(-\infty; 0]$ and non-increasing on $[0; +\infty)$. Let μ be the product measure of μ_1, \dots, μ_n and let $A, B \subset \mathbb{R}^n$ be coordinate boxes containing the origin. Then the function $t \mapsto \mu(A + tB)$ is $\frac{1}{n}$ -concave on \mathbb{R}_+ .*

In the case of non-convex sets, this concavity property is false in general, even for the classical parallel volume $|A + tB_2^n|$ (see Chapter 2).

Bibliography

- [1] L. Ambrosio, A. Colesanti, E. Villa, *Outer Minkowski content for some classes of closed set*, Math. Ann. 342 (2008) no. 4, 727-748.
- [2] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, with a preface by D. Bakry, and M. Ledoux, *Sur les inégalités de Sobolev logarithmiques*, (in French), Panoramas et Synthèses 10, Société Mathématique de France (SMF), Paris, 2000.
- [3] S. Artstein, B. Klartag, V. Milman, *The Santalò point of a function and a functional form of Santalò inequality*, Mathematika 51 (2004), 33-48.
- [4] K. Ball, *Isometric problems in ℓ_p and sections of convex sets*, PhD Dissertation, Cambridge (1986).
- [5] F. Barthe, *On a reverse form of the Brascamp-Lieb inequality*, (English summary) Invent. Math. 134 (1998), no. 2, 335-361.
- [6] W. Beckner, *Inequalities in Fourier analysis on \mathbb{R}^n* , Proc. Nat. Acad. Sci. U.S.A. 72 (1975), 638-641.
- [7] N. M. Blachman, *The convolution inequality for entropy powers*, IEEE Trans. Information theory IT-11 (1965), p. 267-271. 41, 57.
- [8] S. G. Bobkov, *Isoperimetric and analytic inequalities for log-concave probability measures*, Ann. Probab. 27 (1999), no. 4, 1903-1921.
- [9] S. G. Bobkov, I. Gentil, M. Ledoux, *Hypercontractivity of Hamilton-Jacobi equations*, J. Math. Pures Appl. (9) 80 (2001), no. 7, 669-696.
- [10] S. G. Bobkov, M. Ledoux, *From Brunn-Minkowski to Brascamp-Lieb and to logarithmic Sobolev inequalities*, Geom. Funct. Anal. 10 (2000), no. 5, 1028-1052.

-
- [11] S. G. Bobkov, M. Ledoux, *Weighted Poincaré-type inequalities for Cauchy and other convex measures*, Ann. Probab. 37 (2009), no. 2, 403-427.
- [12] S. G. Bobkov, M. Madiman, *Reverse Brunn-Minkowski and reverse entropy power inequalities for convex measures*, J. Funct. Anal. 262 (2012) no. 7, 3309-3339.
- [13] T. Bonnesen, W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934. English translation: *Theory of Convex bodies*, edited by L. Boron, C. Christenson and B. Smith. BCS Associates, Moscow, ID, 1987.
- [14] C. Borell, *Convex measures on locally convex spaces*, Ark.Mat. 12 (1974), 239-252.
- [15] C. Borell, *Convex set functions in d -space*, Periodica Mathematica Hungarica Vol. 6, 111-136, 1975.
- [16] C. Borell, *The Brunn-Minkowski inequality in Gauss space*, Invent. Math. 30 (1975), no. 2, 207-216.
- [17] H. J. Brascamp, E. H. Lieb, *Best constants in Young's inequality, its converse, and its generalization to more than three functions*, Advances in Math. 20 (1976), no. 2, 151-173.
- [18] H. J. Brascamp, E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, (1976) J. Funct. Anal. 22 366-389.
- [19] H. Brunn, *Über Ovale und Eiflächen*, Inaugural Dissertation, München, 1887.
- [20] A. Cianchi, N. Fusco, F. Maggi, A. Pratelli, *On the isoperimetric deficit in Gauss space*, Amer. J. Math. 133 (2011), no. 1, 131-186.
- [21] D. Cordero-Erausquin, M. Fradelizi, B. Maurey, *The (B) conjecture for the Gaussian measure of dilates of symmetric convex sets and related problems*, J. Funct. Anal. 214 (2004), no. 2, 410-427.
- [22] D. Cordero-Erausquin, B. Klartag, *Interpolations, convexity and geometric inequalities*, Geometric aspects of functional analysis, 151-168, Lecture Notes in Math., 2050, Springer, Heidelberg, 2012.
- [23] M. Costa, *A new entropy power inequality*, IEEE Trans. Inform. Theory 31 (1985) 751-760.

-
- [24] M. Costa, T. M. Cover, *On the similarity of the entropy power inequality and the Brunn-Minkowski inequality*, IEEE Trans. Inform. Theory 30 (1984), no. 6, 837-839.
- [25] T. M. Cover, J. A. Thomas, *Elements of Information theory*, Wiley Series in Telecommunications. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1991. xxiv+542 pp. ISBN: 0-471-06259-6.
- [26] S. Dancs, B. Uhrin, *On a class of integral inequalities and their measure-theoretic consequences*, J. Math. Anal. Appl. 74 (1980), no. 2, 388-400.
- [27] Y. De Castro, *Quantitative isoperimetric inequalities on the real line*, Ann. Math. Blaise Pascal 18 (2011), no. 2, 251-271.
- [28] A. Dembo, *Information inequalities and uncertainty principles*, Tech. Rep., Dept. of Statist., Stanford Univ., Stanford, CA, 1990.
- [29] A. Dembo, T. M. Cover, J. A. Thomas, *Information theoretic inequalities*, IEEE Trans. Inform. Theory 37 (1991), no. 6, 1501-1518.
- [30] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 1998. xviii+662 pp.
- [31] H. Federer, *Geometric Measure Theory*, Springer, Berlin, 1969.
- [32] F. Fiala, *Le problème des isopérimètres sur les surfaces ouvertes à courbure positive*, (French) Comment. Math. Helv. 13 (1941) 293-346.
- [33] M. Fradelizi, *Concentration inequalities for s -concave measures of dilations of Borel sets and applications*, Electron. J. Probab. 14 (2009), no. 71, 2068-2090.
- [34] M. Fradelizi, A. Giannopoulos, M. Meyer, *Some inequalities about mixed volumes*, Israel J. Math. 135 (2003), 157-179.
- [35] M. Fradelizi, O. Guédon, *The extreme points of subsets of s -concave probabilities and a geometric localization theorem*, Discrete Comput. Geom. 31 (2004), no. 2, 327-335.
- [36] M. Fradelizi, O. Guédon, *A generalized localization theorem and geometric inequalities for convex bodies*, Adv. Math. 204 (2006), no. 2, 509-529.

-
- [37] M. Fradelizi, A. Marsiglietti, *On the analogue of the concavity of entropy power in the Brunn-Minkowski theory*, Advances in Applied Mathematics 57 (2014), 1-20.
- [38] M. Fradelizi, M. Meyer, *Some functional forms of Blaschke-Santalò inequality*, Math. Z. 256 (2007), no. 2, 379-395.
- [39] M. Fradelizi, M. Meyer, *Functional inequalities related to Mahler's conjecture*, Monatsh. Math. 159 (2010), no. 1-2, 13-25.
- [40] R. J. Gardner, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N.S.) 39 (2002), no. 3, 355-405.
- [41] R. J. Gardner, A. Zvavitch, *Gaussian Brunn-Minkowski inequalities*, Trans. Amer. Math. Soc. 362 (2010), no. 10, 5333-5353.
- [42] I. Gorbovickis, *The strict Kneser-Poulsen conjecture for large radii*, Geom. Dedicata vol. 162 no. 1 (2013) 95-107.
- [43] N. Gozlan, C. Roberto, P-M. Samson, *Hamilton Jacobi equations on metric spaces and transport-entropy inequalities*, Revista Matematica Iberoamericana, 30 (2014), no. 1, 133-163.
- [44] O. Guédon, *Kahane-Khinchine type inequalities for negative exponent*, Mathematika 46 (1999), no. 1, 165-173.
- [45] O. Guleryuz, E. Lutwak, D. Yang, G. Zhang, *Information-theoretic inequalities for contoured probability distributions*, IEEE Trans. Inform. Theory 48 (2002) no. 8, 2377-2383.
- [46] G. H. Hardy, J. E. Littlewood, G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1959.
- [47] R. V. L. Hartley, *Transmission of Information*, Bell System Technical Journal, July 1928, pp. 535-563.
- [48] R. Henstock, A. M. Macbeath, *On the measure of sum-sets. I. The theorems of Brunn, Minkowski, and Lusternik*, Proc. London Math. Soc. (3) 3, (1953). 182-194.
- [49] M. A. Hernández Cifre, J. Yepes Nicolás, *Refinements of the Brunn-Minkowski inequality*, J. Convex Anal. 21 (3) (2014), 1-17.

-
- [50] O. Hölder, *Ueber einen Mittelwerthssatz*, Nachrichten von der Königl. Gesellschaft der Wissenschaften und der Georg-Augusts-Universität zu Göttingen, 1889, 38-47.
- [51] D. Hug, G. Last, W. Weil, *A local Steiner-type formula for general closed sets and applications*, Math. Z. 246 (2004), no. 1-2, 237-272.
- [52] O. Johnson, *Information theory and the Central Limit Theorem*, Imperial College Press, London, 2004.
- [53] J. Kampf, *The parallel volume at large distances*, Geom. Dedicata, 160 (2012) 47-70.
- [54] J. Kampf, *Asymptotic order of the parallel volume difference*, WiMa Report 139. TU Kaiserslautern, preprint.
- [55] J. Kampf, M. Kiderlen *Large parallel volumes of finite and compact sets in d -dimensional Euclidean space*, Doc. Math. 18 (2013), 275-295.
- [56] R. Kannan, L. Lovász, M. Simonovits, *Isoperimetric problems for convex bodies and a localization lemma*, (English summary) Discrete Comput. Geom. 13 (1995), no. 3-4, 541-559.
- [57] A. V. Kolesnikov, E. Milman, *Poincaré and Brunn-Minkowski inequalities on weighted Riemannian manifolds with boundary*, preprint, arXiv:1310.2526 [math.DG]
- [58] R. Latała, *On some inequalities for Gaussian measures*, (English summary) Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 813-822, Higher Ed. Press, Beijing, 2002.
- [59] J. Lehec, *A direct proof of the functional Santalò inequality*, C. R. Math. Acad. Sci. Paris 347 (2009), no. 1-2, 55-58.
- [60] L. Leindler, *On a certain converse of Hölder's inequality*, II, Acta Sci. Math., 33 (1972), 217-223.
- [61] A. Livne Bar-on, *The (B) conjecture for uniform measures in the plane*, Geometric Aspects of Functional Analysis, Lecture Notes in Mathematics Volume 2116, 2014, 341-353.
- [62] L. Lovász, M. Simonovits, *Random walks in a convex body and an improved volume algorithm*, Random Structures Algorithms 4 (1993), no. 4, 359-412.

- [63] L. A. Lusternik, *Die Brunn-Minkowskische Ungleichnung für beliebige messbare Mengen*, Comptes Rendus (Doklady) de l'académie des Sciences de l'URSS (Nouvelle Série) III: 55-58, 1935.
- [64] A. Marsiglietti, *Concavity properties of extensions of the parallel volume*, *Mathematika* (2015), doi:10.1112/S0025579314000369.
- [65] A. Marsiglietti, *On the improvement of concavity of convex measures*, *Proc. Amer. Math. Soc.* (2015), doi: <http://dx.doi.org/10.1090/proc/12694>.
- [66] H. Minkowski, *Geometrie der Zahlen*, Leipzig, Teubner, 1896.
- [67] P. Nayar, T. Tkocz, *A note on a Brunn-Minkowski inequality for the Gaussian measure*, *Proc. Amer. Math. Soc.* 141 (2013), no. 11, 4027-4030.
- [68] V. H. Nguyen, *Dimensional variance inequalities of Brascamp-Lieb type and a local approach to dimensional Prékopa's theorem*, *J. Funct. Anal.* 266 (2014), no. 2, 931-955.
- [69] H. Nyquist, *Certain topics in telegraph transmission theory*, *Trans. AIEE*, vol. 47, pp. 617-644, Apr. 1928.
- [70] A. Prékopa, *Logarithmic concave measures with application to stochastic programming*, *Acta Sci. Math.*, 32 (1971), 301-316.
- [71] A. Prékopa, *On logarithmic concave measures and functions*, *Acta Sci. Math.* (Szeged) 34 (1973), 335-343.
- [72] J. Rataj, V. Schmidt, E. Spodarev, *On the expected surface area of the Wiener sausage*, *Math. Nachr.* 282 (2009), no. 4, 591-603.
- [73] L. J. Rogers, *An extension of a certain theorem in inequalities*, *The Messenger of Mathematics*, Vol. xvii no. 10. (1888), 145-150.
- [74] C. Saroglou, *Remarks on the conjectured log-Brunn-Minkowski inequality*, *Geom. Dedicata* 177 (2015), 353-365.
- [75] G. Savaré, G. Toscani, *The concavity of Rényi entropy power*, *IEEE Trans. Inform. Theory* 60 (2014), no. 5, 2687-2693.

- [76] R. Schneider, *A measure of convexity for compact sets*, Pacific J. Math. 58 (1975), no. 2, 617-625.
- [77] R. Schneider, *Convex bodies : the Brunn-Minkowski theory*, Encyclopedia of mathematics and its applications 44, Cambridge University Press, Cambridge, 1993.
- [78] C. E. Shannon, *A mathematical theory of communication*, Bell System Tech. J. 27, (1948). 379-423, 623-656.
- [79] L. Stachó, *On the volume function of parallel sets*, Acta Sci. Math. 38 (1976), no. 3-4, 365-374.
- [80] A. Stam, *Some inequalities satisfied by the quantities of information of Fisher and Shannon*, Information and Control 2 (1959), p. 101-112.
- [81] J. Steiner, *Über parallele Flächen*, Monatsbericht der Akademie der Wissenschaften zu Berlin (1840), pp. 114-118.
- [82] V. N. Sudakov, B. S. Cirel'son, *Extremal properties of half-spaces for spherically invariant measures* (Russian) Problems in the theory of probability distributions, II. Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 41 (1974), 14-24, 165.
- [83] S. J. Szarek and D. Voiculescu, *Shannon's entropy power inequality via restricted Minkowski sums*, Geometric aspects of functional analysis in Lecture Notes in Math., 1745, Springer, Berlin, 2000, 257-262.
- [84] C. Villani, *A short proof of the "concavity of entropy power"*, IEEE Trans. Inform. Theory 46 (2000) no. 4, 1695-1696.
- [85] W. H. Young, *On the Determination of the Summability of a Function by Means of its Fourier Constants*, Proc. London Math. Soc. (1913) S2-12 no. 1, 71.

Dictionary

The Brunn-Minkowski theory

Definition 4.2.11 (γ -concave function). Let $\gamma \in [-\infty, +\infty]$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ be a non-negative function. One says that f is γ -concave if for every $\lambda \in [0, 1]$ and for every $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$, one has

$$f((1 - \lambda)x + \lambda y) \geq ((1 - \lambda)f(x)^\gamma + \lambda f(y)^\gamma)^{\frac{1}{\gamma}}.$$

The limit cases are interpreted by continuity, i.e. the right-hand side of this inequality is equal to $f(x)^{1-\lambda}g(y)^\lambda$ for $s = 0$, to $\min(f(x), g(y))$ for $s = -\infty$ and to $\max(f(x), g(y))$ for $s = +\infty$. Notice that a 1-concave function is concave on its support, that a $-\infty$ -concave function has convex level sets $\{x; f(x) \geq t\}$, and that a $+\infty$ -concave function is constant on its support.

Definition 4.2.12 (s -concave measure). Let $s \in [-\infty, +\infty]$ and μ be a non-negative measure. One says that μ is s -concave if for every $\lambda \in [0, 1]$ and for every compact set $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$, one has

$$\mu((1 - \lambda)A + \lambda B) \geq ((1 - \lambda)\mu(A)^s + \lambda\mu(B)^s)^{\frac{1}{s}}.$$

The limit cases are interpreted by continuity, i.e. the right-hand side of this inequality is equal to $\mu(A)^{1-\lambda}\mu(B)^\lambda$ for $s = 0$, which corresponds to log-concave measures, to $\min(\mu(A), \mu(B))$ for $s = -\infty$ and to $\max(\mu(A), \mu(B))$ for $s = +\infty$.

Definition 4.2.13 (Cone). A set A is a cone if for every $x \in A$ and every $\lambda \geq 0$, then $\lambda x \in A$.

Definition 4.2.14 (Convex body). A set K is a convex body if K is a compact convex set with non-empty interior.

Definition 4.2.15 (Convex combination). A point $x \in \mathbb{R}^n$ is a convex combination of $x_1, \dots, x_k \in \mathbb{R}^n$ if there exists non-negative real $\lambda_1, \dots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i = 1$ and $x = \sum_{i=1}^k \lambda_i x_i$.

Definition 4.2.16 (Convex hull). Let A be a subset of \mathbb{R}^n . The convex hull of A is the set of all convex combinations of points of A . It is also the intersection of all convex sets containing A . It is also the smallest convex set containing A .

Definition 4.2.17 (Convex set). A set C is convex if for every $x, y \in C$ and for every $\lambda \in [0, 1]$, then $(1 - \lambda)x + \lambda y \in C$.

Definition 4.2.18 (Dilation). Let A be a subset of \mathbb{R}^n and $\lambda \in \mathbb{R}$. The dilation of A by λ is

$$\lambda A = \{\lambda a; a \in A\}.$$

Definition 4.2.19 (Halfspace). Let H be a hyperplane. The halfspaces determined by H are the sets

$$H^+ = \{x \in \mathbb{R}^n; \langle \theta, x \rangle \geq \alpha\} \text{ and } H^- = \{x \in \mathbb{R}^n; \langle \theta, x \rangle \leq \alpha\}.$$

Definition 4.2.20 (Hyperplane). A hyperplane H of \mathbb{R}^n is an affine subspace of dimension $n - 1$. It is the set

$$H = \{x \in \mathbb{R}^n; \langle x, \theta \rangle = \alpha\}$$

for some $\theta \in \mathcal{S}^{n-1}$ and $\alpha \in \mathbb{R}$.

Definition 4.2.21 (Minkowski sum). Let A and B be two subsets of \mathbb{R}^n . The Minkowski sum of A and B is

$$A + B = \{a + b; a \in A, b \in B\}.$$

Definition 4.2.22 (Norm). A function $|\cdot| : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a norm if

1. For every $x \in \mathbb{R}^n$, if $|x| = 0$, then $x = 0$.
2. For every $x \in \mathbb{R}^n$ and for every $\lambda \geq 0$, one has $|\lambda x| = \lambda |x|$.
3. For every $x, y \in \mathbb{R}^n$, one has $|x + y| \leq |x| + |y|$.

Definition 4.2.23 (Orthogonal of a set). Let A be a subset of \mathbb{R}^n . The orthogonal of A is the set

$$\{x \in \mathbb{R}^n; \forall y \in A, \langle x, y \rangle = 0\}.$$

Definition 4.2.24 (Orthogonal projection on a set). *Let A and K be two subsets of \mathbb{R}^n . The orthogonal projection of K on A is the set*

$$\{x \in A; \exists u \in A^\perp, x + u \in K\}.$$

Definition 4.2.25 (Parallel set). *Let A be a subset of \mathbb{R}^n . The parallel set of A at distance $t \geq 0$ is the set $A + tB_2^n$.*

Definition 4.2.26 (Parallel volume). *Let A be a measurable subset of \mathbb{R}^n . The parallel volume of A is the volume of the parallel set of A , namely $|A + tB_2^n|$, $t \in \mathbb{R}_+$.*

Definition 4.2.27 (Parallel μ -volume). *Let μ be a non-negative measure. Let A be a subset of \mathbb{R}^n . The parallel μ -volume of A is the measure, with respect to μ , of the parallel set of A , namely $\mu(A + tB_2^n)$, $t \in \mathbb{R}_+$.*

Definition 4.2.28 (Polytope). *A polytope is the convex hull of a finite number of points.*

Definition 4.2.29 (Polyhedron). *A polyhedron is a finite intersection of halfspaces.*

Definition 4.2.30 (Semi-norm). *A function $p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is a semi-norm if*

1. *For every $x \in \mathbb{R}^n$ and for every $\lambda \geq 0$, one has $p(\lambda x) = \lambda p(x)$.*
2. *For every $x, y \in \mathbb{R}^n$, one has $p(x + y) \leq p(x) + p(y)$.*

Definition 4.2.31 (Simplex). *A simplex in \mathbb{R}^n is the convex hull of $(n + 1)$ points not all contained in a strict affine subspace of \mathbb{R}^n .*

Definition 4.2.32 (Support hyperplane). *Let A be a subset of \mathbb{R}^n . A hyperplane H is a support hyperplane of C if there exists $x \in C \cap H$ and such that $C \subset H^+$ or $C \subset H^-$.*

The Information theory

Definition 4.2.33 (Entropy - Discrete case). *Let X be a discrete random variable having values x_1, \dots, x_n with probabilities p_1, \dots, p_n respectively. The entropy of X is*

$$H(X) = - \sum_{i=1}^n p_i \log(p_i),$$

with the convention that $0 \log(0) = 0$.

Definition 4.2.34 (Entropy - Continuous case). *Let X be a random vector in \mathbb{R}^n with probability density function f . The entropy of X is*

$$H(X) = - \int f \log(f).$$

Definition 4.2.35 (Entropy power). *Let X be a continuous random vector in \mathbb{R}^n . The entropy power of X is*

$$N(X) = \frac{1}{2\pi e} e^{2H(X)},$$

where $H(X)$ is the entropy of X .

Definition 4.2.36 (Fisher information). *Let X be a continuous random vector in \mathbb{R}^n with score function ρ_X . The Fisher information of X is*

$$I(X) = \mathbb{E}_X[|\rho_X|^2] = \int_{\mathbb{R}^n} \frac{|\nabla f(x)|^2}{f(x)} dx.$$

Definition 4.2.37 (Relative entropy - Discrete case). *Let X and Y be two random variables with probabilities of realization p_1, \dots, p_n and q_1, \dots, q_n respectively. The relative entropy of X with respect to Y is*

$$D(X||Y) = \sum_{i=1}^n p_i \log \left(\frac{p_i}{q_i} \right).$$

Definition 4.2.38 (Relative entropy - Continuous case). *Let X and Y be two random vectors with probability densities f and g respectively. The relative entropy of X with respect to Y is*

$$D(X||Y) = \int f \log \left(\frac{f}{g} \right).$$

Definition 4.2.39 (*p*-th Rényi entropy). Let $p \in (0, 1) \cup (1, +\infty)$. Let X be a continuous random vector in \mathbb{R}^n . The *p*-th Rényi entropy of X is

$$H_p(X) = \frac{1}{1-p} \log \left(\int_{\mathbb{R}^n} f^p(x) \, dx \right).$$

Definition 4.2.40 (Score function). Let X be a continuous random vector in \mathbb{R}^n with probability density f . The score function of X is

$$\rho_X(x) = \frac{\nabla f(x)}{f(x)}, \quad x \in \mathbb{R}^n.$$

Notations

\mathbb{N} : Set of non-negative integers
 \mathbb{R} : Set of real numbers
 \mathbb{R}_+ : Set of non-negative real numbers
 \mathbb{R}^n : Set of n -tuples with value in \mathbb{R}
 $\{e_1, \dots, e_n\}$: Canonical basis of \mathbb{R}^n
 $\langle \cdot, \cdot \rangle$: Scalar product
 $|x|$: Euclidean norm of $x \in \mathbb{R}^n$
 B_2^n : Euclidean closed unit ball
 S^{n-1} : Euclidean closed unit sphere
 $d(x, A)$: Distance from the point x to the set A
 \in : Belong to
 \subset : Included in
 A, B : Subsets of \mathbb{R}^n
 K, L : Convex bodies of \mathbb{R}^n
 $A + B$: Minkowski sum of A and B
 λA : Dilates of A
 $|A|$: Volume of A
 V_A : Parallel volume of A
 1_A : Indicator function of A
 A^\perp : Orthogonal of A
 $\text{Aff}(A)$: Affine space generated by A
 ∂A : Boundary of A
 $\text{conv}(A)$: Convex hull of A
 $\text{diam}(A)$: Diameter of A
 $\text{card}(A)$: Cardinal of A
 $\text{int}(A)$: Interior of A
 \bar{A} : closure of A

\angle : Angle
 $\widehat{}$: Arc
 f, g, h : Functions
 $f \star g$: Convolution of f and g
 f'_+ : Right derivative of f
 f'_- : Left derivative of f
 $\frac{\partial}{\partial x}$: Partial derivative
 ∇f : Gradient of f
 Δ : Laplace operator
 $Hess(f)$: Hessian of f
 L^p : Set of p -integrable functions
 $\|f\|_p$: L^p norm of f
 $\text{supp}(f)$: Support of f
 $\lim f_n$: Limit of the sequence (f_n)
 $\det(M)$: Determinant of M
 $HessM$: Hessian of M
 X, Y : Random vectors in \mathbb{R}^n
 $\mathbb{E}[X]$: Expectation of X
 $Var(X)$: Variance of X
 $H(X)$: Entropy of X
 $N(X)$: entropy power of X
 $H_p(X)$: p -th Rényi entropy of X
 $I(X)$: Fisher information of X
 ρ_X : Score function of X
 $D(X||Y)$: Relative entropy of X with respect to Y
 \log : Logarithm function
 e : Exponential function
 μ : non-negative measure
 \mathcal{H}^n : n -dimensional Hausdorff measure