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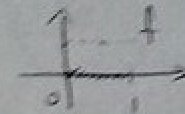
Review - Lebesgue Integration

<u>Q:</u> What does $\int_a^b f(x)\,dx$ mean?

→ Area under the curve? But what is an area?

→ what if $f(x) = \mathbb{1}_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \text{ rational} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$



- Advantage of Lebesgue integral (compared to Riemann integral):

  a) $f(x) = \mathbb{1}_{\mathbb{Q}}(x)$ is Lebesgue integrable (but not Riemann integrable)

  b) Conditions for $\lim_n \int f_n = \int \lim_n f_n$ much weaker for Lebesgue integral (Riemann integral require uniform convergence...)

  c) Lebesgue integral allows measures.

<u>Setting:</u> $(\Omega, \mathcal{F}, P)$ measure space (probability space if P is a probability measure)

<u>Def:</u> An application $X : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a <u>random variable</u> if:

$$X^{-1}(\mathcal{B}(\mathbb{R})) \subset \mathcal{F}.$$

$$\iff \forall B \in \mathcal{B}(\mathbb{R}),\ X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

<u>Notation:</u> $X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} = \{X \in B\}$.

Measure theory	Probability theory
set E	Sample space Ω
σ-algebra $\mathcal{A}$	Event space $\mathcal{F}$
measure μ	probability measure P
element $x \in E$	outcome $\omega \in \Omega$
measurable function f	random variable X
Lebesgue integral $\int f\,d\mu$	Expectation $\mathbb{E}[X] = \int_\Omega X(\omega)\,dP(\omega)$.