Construction of Lebesgue integral:

Def: $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a simple random variable if

$$\exists\, a_1, \dots, a_n \in \mathbb{R}, \quad A_1, \dots, A_n \in \mathcal{F} \text{ s.t.}$$

$$X = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}.$$

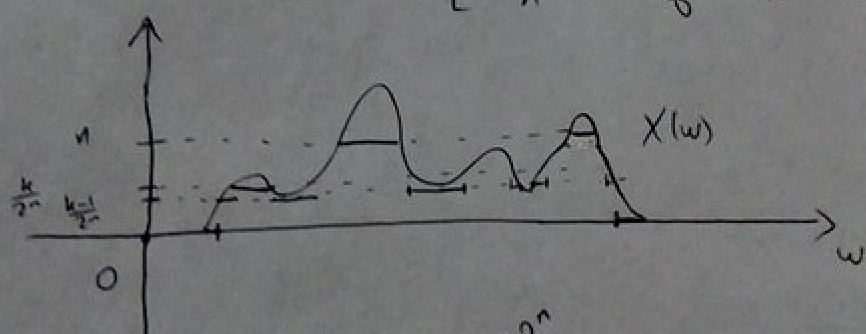
Main Lemma: Let X be a non-negative random variable, then $\exists \{X_n\}$ increasing sequence of non-negative simple random variables such that

$$\forall \omega \in \Omega, \quad X(\omega) = \lim_n X_n(\omega)$$

(Everywhere convergence)

Pf: For $n \geq 1$ and $\omega \in \Omega$, define

$$X_n(\omega) = \begin{cases} \frac{k-1}{2^n} & \text{if } \frac{k-1}{2^n} \leq X(\omega) < \frac{k}{2^n} \quad (k \in \{1, \dots, n2^n\}) \\ n & \text{if } X(\omega) \geq n \end{cases}$$



One can write $$X_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbb{1}_{\{\frac{k-1}{2^n} \leq X < \frac{k}{2^n}\}} + n \mathbb{1}_{\{X \geq n\}}.$$

Def: Let $X = \sum_{k=1}^{n} a_k \mathbb{1}_{A_k}$ be a simple r.v.

The Lebesgue integral of X with respect to the measure P is:

$$\int_\Omega X(\omega)\, dP(\omega) \overset{\text{notation}}{=} \sum_{k=1}^{n} a_k P(A_k).$$

Rem: This definition does not depend on the representation of X. (Exercise)

Notation: $\int X\, dP = \int_\Omega X(\omega)\, dP(\omega) = \mathbb{E}[X] = \sum_{k=1}^{n} a_k P(A_k).$