(2) <u>Rem:</u> $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$.

<u>Def:</u> Let X be a non-negative random variable. Then,

$$\mathbb{E}[X] = \sup \{ \mathbb{E}[Y] : Y \text{ simple random variable}, \ Y \leq X \}$$

<u>Def:</u> Let X be a random variable. We say that X is $\mathbb{P}$-integrable if

$$\mathbb{E}[X^+] < +\infty \text{ and } \mathbb{E}[X^-] < +\infty.$$

In this case, $\mathbb{E}[X] = \mathbb{E}[X^+] - \mathbb{E}[X^-]$,

where $X^+ = \max(X, 0)$, $X^- = \max(-X, 0)$.

<u>Rem:</u> a) $X^+, X^- \geq 0$

b) $X = X^+ - X^-$

c) $|X| = X^+ + X^-$.

<u>Approx:</u> Let $\omega \in \Omega$. If $X(\omega) = +\infty$, then $X_n(\omega) = n$, $\forall n$.

$\Rightarrow \lim_{n\to+\infty} X_n(\omega) = +\infty = X(\omega)$

If $X(\omega) < +\infty$. Then, for $n > X(\omega)$,

$$0 \leq |X_n(\omega) - X(\omega)| = X(\omega) - X_n(\omega) \leq \frac{1}{2^n} \xrightarrow[n\to+\infty]{} 0$$

<u>Convergence Theorems:</u>

<u>Th:</u> (Monotone Convergence, or Beppo Levi)

Let $\{X_n\}$ <u>increasing</u> sequence of <u>non-negative</u> r.v.

Then, $\exists X$ r.v. s.t. $X_n \xrightarrow[n\to+\infty]{} X$ a.s. and

$$\lim_{n\to+\infty} \mathbb{E}[X_n] = \mathbb{E}[X]. \quad (\text{limit is possibly } +\infty)$$

<u>Coro:</u> For $X \geq 0$. $\mathbb{E}[X] = \lim_{n\to+\infty} \mathbb{E}[X_n]$, where $\{X_n\}$ sequence of simple r.v. approximating X.

<u>Th:</u> (Fatou lemma)

Let $\{X_n\}$ sequence of <u>non-negative</u> r.v. Then

$$\mathbb{E}[\underline{\lim}\, X_n] \leq \underline{\lim}\, \mathbb{E}[X_n].$$