Coro: If $X_n \to X$ a.s and $\{X_n\}$ bounded in L^1
($\exists M, \forall n \geq 1, \mathbb{E}[|X_n|] \leq M$)

Then $\mathbb{E}[|X|] \leq M$.

Pf: $\mathbb{E}[|X|] = \mathbb{E}[\lim_{n \to +\infty} |X_n|] = \mathbb{E}[\underline{\lim} |X_n|] \leq \underline{\lim}\, \mathbb{E}[|X_n|] \leq M$.

Th: (Lebesgue dominated convergence)
$\{X_n\}$ sequence of integrable r.v. such that
a) $X_n \to X$ a.s.
b) $\exists Y$ $\underbrace{\text{r.v.}}_{\text{in } L^1\ (\mathbb{E}[|Y|] < +\infty)}$ st $\forall n \geq 1$, $|X_n| \leq Y$.

Then X is in L^1 ($\mathbb{E}[|X|] < +\infty$) and

$$\mathbb{E}[\lim_{n \to +\infty} X_n] = \mathbb{E}[X] = \lim_{n \to +\infty} \mathbb{E}[X_n].$$

L^p-space: ($p \geq 1$)
$(\Omega, \mathcal{F}, P)$ probability space.

$$L^p(\Omega, \mathcal{F}, P) = \{ X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ random variable such that } \mathbb{E}[|X|^p] < +\infty \}.$$

Th: a) $\|X\|_{L^p} = \mathbb{E}[|X|^p]^{\frac{1}{p}}$ is a norm on $L^p(\Omega, \mathcal{F}, P)$.
b) $(L^p, \|\cdot\|_p)$ is a Banach space (Cauchy sequences converge).

Def: $X_n \to X$ in L^p if $\|X_n - X\|_p \xrightarrow[n \to +\infty]{} 0$

$$\iff \mathbb{E}[|X_n - X|^p] \xrightarrow[n \to +\infty]{} 0.$$

Th: $(L^2, \langle X, Y \rangle)$, where $\langle X, Y \rangle = \mathbb{E}[XY]$, is a Hilbert space.