3

**Prop:** If $p \le q$, then $L^q \subset L^p$.

**Rem:** ⚠ This is true because $P$ is a finite measure. (False otherwise).

**Th:** (Fubini)

$(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ proba space.

Let $\Omega = \Omega_1 \times \Omega_2$, $\mathcal{F} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$, $P = P_1 \otimes P_2$. Let $X$ r.v. on $(\Omega, \mathcal{F})$

If $X$ is $\geq 0$ or integrable, then

$$\int_\Omega X(\omega)\, dP(\omega) = \int_{\Omega_1} \left[ \int_{\Omega_2} X(\omega_1, \omega_2)\, dP_2(\omega_2) \right] dP_1(\omega_1)$$

$$= \int_{\Omega_2} \left[ \int_{\Omega_1} X(\omega_1, \omega_2)\, dP_1(\omega_1) \right] dP_2(\omega_2).$$

**Application:** $(p \geq 1)$

$$\mathbb{E}[|X|^p] = \mathbb{E}\left[\int_0^{|X|} p t^{p-1} dt\right] = \mathbb{E}\left[\int_0^{+\infty} p t^{p-1} \mathbb{1}_{\{|X|>t\}} dt\right]$$

$$\overset{\text{Fubini}}{=} \int_0^{+\infty} p t^{p-1}\, \mathbb{E}[\mathbb{1}_{\{|X|>t\}}]\, dt$$

$$= \int_0^{+\infty} p t^{p-1}\, P(|X|>t)\, dt$$

. For $p=1 \Rightarrow \mathbb{E}[|X|] = \int_0^{+\infty} P(|X|>t)\, dt.$