Conditional Expectation

Def - Th: $(\Omega, \mathcal{F}, \mathbb{P})$ probability space. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra.
Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$.

The <u>conditional expectation</u> of X with respect to $\mathcal{G}$, denoted by $\mathbb{E}[X|\mathcal{G}]$, is a random variable Y such that:

- Y is $\mathcal{G}$-measurable
- $\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[Y \mathbb{1}_A]$, $\forall A \in \mathcal{G}$.

Rem: $\forall A \in \mathcal{G}$, $\mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}] \mathbb{1}_A]$, and $\mathbb{E}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable.

Prop: i) $\forall X, Y \in L^1(\mathcal{F})$, $\forall \lambda \in \mathbb{R}$, $\mathbb{E}[\lambda X + Y | \mathcal{G}] = \lambda \mathbb{E}[X|\mathcal{G}] + \mathbb{E}[Y|\mathcal{G}]$

ii) $\forall X \in L^1(\mathcal{F})$, $|\mathbb{E}[X|\mathcal{G}]| \leq \mathbb{E}[|X| \,|\, \mathcal{G}]$.

iii) $\forall X \in L^1(\mathcal{F})$, $\mathbb{E}[\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[X]$.

iv) (Tower property) Let $\mathcal{G}_2 \subset \mathcal{G}_1 \subset \mathcal{F}$. Let $X \in L^1(\mathcal{F})$. Then,

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}_1]|\mathcal{G}_2] = \mathbb{E}[X|\mathcal{G}_2].$$

Pf: i) Let $A \in \mathcal{G}$.

$$\mathbb{E}[(\lambda X + Y)\mathbb{1}_A] = \lambda \mathbb{E}[X \mathbb{1}_A] + \mathbb{E}[Y \mathbb{1}_A] = \lambda \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\mathbb{1}_A] + \mathbb{E}[\mathbb{E}[Y|\mathcal{G}]\mathbb{1}_A]$$

$$= \mathbb{E}[\underbrace{(\lambda \mathbb{E}[X|\mathcal{G}] + \mathbb{E}[Y|\mathcal{G}])}_{\mathcal{G}\text{-meas.}} \mathbb{1}_A]$$