ii) $X = X_+ - X_-$ where $X_+ = \max(X,0)$ and $X_- = \max(-X,0)$.

$$\Rightarrow |\mathbb{E}[X|\mathcal{G}]| = |\mathbb{E}[X_+|\mathcal{G}] - \mathbb{E}[X_-|\mathcal{G}]|$$
$$\leq |\mathbb{E}[X_+|\mathcal{G}]| + |\mathbb{E}[X_-|\mathcal{G}]|$$
$$= \mathbb{E}[X_+|\mathcal{G}] + \mathbb{E}[X_-|\mathcal{G}]$$
$$= \mathbb{E}[X_+ + X_-|\mathcal{G}] = \mathbb{E}[|X| \,|\mathcal{G}]$$

Fact: If $X \geq 0$, then $\mathbb{E}[X|\mathcal{G}] \geq 0$.

iii) Take $A = \Omega$.

iv) (e.x.)

- Conditional expectation and Independence: $(\Omega, \mathcal{F}, P)$ proba space.

Def: a) $A, B \in \mathcal{F}$ independents $\Leftrightarrow P(A \cap B) = P(A)P(B)$.

b) $\mathcal{A}, \mathcal{B} \subset \mathcal{F}$ sub-σ-algebras are independents $\Leftrightarrow \forall A \in \mathcal{A}, \forall B \in \mathcal{B}$ $P(A \cap B) = P(A)P(B)$.

c) X, Y independents $\Leftrightarrow \sigma(X)$ and $\sigma(Y)$ are independents

$\Leftrightarrow \forall I \in \mathcal{B}(\mathbb{R}), \forall J \in \mathcal{B}(\mathbb{R})$
$P(X \in I, Y \in J) = P(X \in I)P(Y \in J)$.

$\Leftrightarrow \mathbb{E}[\varphi(X)\psi(Y)] = \mathbb{E}[\varphi(X)]\mathbb{E}[\psi(Y)]$, $\forall \varphi, \psi : \mathbb{R} \to \mathbb{R}$ meas, bounded

Prop: a) X independent of $\mathcal{G}$ and Y $\mathcal{G}$-measurable $\Rightarrow X \perp\!\!\!\perp Y$.

b) X independent of $\mathcal{G} \Rightarrow \varphi(X)$ independent of $\mathcal{G}$, $\forall \varphi : \mathbb{R} \to \mathbb{R}$ meas.

Th: If $X \perp\!\!\!\perp \mathcal{G}$, then $\mathbb{E}[X|\mathcal{G}] = \mathbb{E}[X]$.

Pf: Let $A \in \mathcal{G}$. $\mathbb{E}[X \mathbb{1}_A] \underset{X \perp \mathbb{1}_A}{=} \mathbb{E}[X]\mathbb{E}[\mathbb{1}_A] = \mathbb{E}[\underbrace{\mathbb{E}[X]\mathbb{1}_A}_{\mathcal{G}\text{-meas.}}]$.

Th: $X \perp\!\!\!\perp \mathcal{G} \Leftrightarrow \mathbb{E}[\phi(X)|\mathcal{G}] = \mathbb{E}[\phi(X)]$, $\forall \phi : \mathbb{R} \to \mathbb{R}$ meas.