University of Florida **MAA 4102/5104**

Previous Homework

PARTIAL SOLUTIONS

Homework #3

Exercise 10.

False. Counterexample: Consider, for $n \geq 1$,

$$a_n = \frac{1}{n}, \quad b_n = n.$$

Then, $\{a_n\}$ converges to 0, but the product sequence $\{a_n b_n\}$ converges to 1 (because, for all $n \geq 1$, $a_n b_n = 1$).

Exercise 11.

Consider, for $n \in \mathbb{N}$,

$$a_n = n + (-1)^n.$$

Exercise 12.

Consider, for $n \geq 1$,

$$a_n = 1 - \frac{1}{n}.$$

Exercise 13.

4. For all $n \geq 0$, we have

$$\frac{a_{n+1}}{a_n} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2^{n+1}(n+1)!} \frac{2^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} = \frac{2n+1}{2(n+1)} = \frac{2n+1}{2n+2} \leq 1.$$

Since $a_n \geq 0$, we deduce that $a_{n+1} \leq a_n$. Hence, $\{a_n\}$ is decreasing. Since $\{a_n\}$ is decreasing and lower bounded (by 0), a theorem in class allows us to conclude that $\{a_n\}$ converges.

Homework #4

Exercise 3.

A theorem in class tells us that a sequence is Cauchy if and only if it is convergent (as we work with sequences in $\mathbb{R}$).

- 1., 2., 3., 5. are all convergent sequences, so they are Cauchy.
- 4. is not a Cauchy sequence (done in class).
- 6. Yes, it is a Cauchy sequence: Let $m \leq n$. Then, using triangle inequality,

$$|a_n - a_m| = |a_n - a_{n-1} + \cdots + a_{m+1} - a_m| \leq |a_n - a_{n-1}| + \cdots + |a_{m+1} - a_m|$$

$$\leq r^{n-1} + \cdots + r^m = r^m(1 + \cdots + r^{n-m-1}) \leq r^m \sum_{k=0}^{+\infty} r^k = \frac{r^m}{1-r},$$

which converges to 0 as $n, m \to +\infty$.

Exercise 7.

1. $a_{2n} = \frac{1-1}{2} = 0$, and $a_{2n+1} = \frac{1+1}{2} = 1$. The subsequential limits are 0 and 1.

2. $a_{2n} = \sin(n\pi) = 0$, and $a_{4n+1} = \sin(2n\pi + \frac{\pi}{2}) = 1$, and $a_{4n+3} = \sin(2n\pi + \frac{3\pi}{2}) = -1$. The subsequential limits are -1,0,1.

3. $a_{2n} = \frac{2n-1}{2n}$, which converges to 1, and $a_{2n+1} = -\frac{2n}{2n+1}$, which converges to -1. The subsequential limits are -1 and 1.

• The above sequences diverge because of the following theorem:

A sequence $\{a_n\}$ converges to L if and only if all its subsequences converge to L.

Homework #5

Exercise 1.

2. We have for all $x > 0$,

$$\frac{\sqrt{x}}{x^2 - 1} = \frac{\sqrt{x}}{x^2} \frac{1}{1 - \frac{1}{x^2}} = \frac{1}{x^{3/2}} \frac{1}{1 - \frac{1}{x^2}}.$$

Since $\lim_{x \to +\infty} \frac{1}{x^2} = 0$, we have, using theorems on limits, $\lim_{x \to +\infty} \frac{1}{1 - \frac{1}{x^2}} = 1$. And since $\lim_{x \to +\infty} \frac{1}{x^{3/2}} = 0$, we have, using theorems on limits,

$$\lim_{x \to +\infty} \frac{1}{x^{3/2}} \frac{1}{1 - \frac{1}{x^2}} = \left(\lim_{x \to +\infty} \frac{1}{x^{3/2}} \right) \left(\lim_{x \to +\infty} \frac{1}{1 - \frac{1}{x^2}} \right) = 0.$$

3. We have for all $x > 0$,

$$\left| \frac{\sin(x)}{x} \right| \leq \frac{1}{x}.$$

Since $\lim_{x \to +\infty} \frac{1}{x} = 0$, by squeeze theorem we conclude that $\lim_{x \to +\infty} \frac{\sin(x)}{x} = 0$.

Exercise 4.

We use the following theorem from class:

• $\lim_{x \to a} f(x) = L$ if and only if for all sequence $\{x_n\}$ that converges to a, the sequence $\{f(x_n)\}$ converges to L.

Now, denote $f(x) = \frac{1}{x}$, and consider the following sequences: $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$. Then, x_n and y_n converges to 0, but $f(x_n) = n$ converges to $+\infty$ and $f(y_n) = -n$ converges to $-\infty$. Hence, $\lim_{x \to 0} \frac{1}{x}$ does not exist.

Exercise 5.

Same argument as Exercise 4. For example, consider $x_n = \frac{1}{2\pi n}$ and $y_n = \frac{1}{\frac{\pi}{2} + 2\pi n}$.