Homework #10 – **Higher-order derivatives**

PARTIAL SOLUTIONS

It is assumed that you know how to compute the derivative of most encountered functions, such as polynomials, quotient of polynomials, exponential, logarithm, and trigonometric functions, as well as a composition of these functions.

Exercise 1.

Proof by induction.

- If $n = 1$, then $x^n = x$ and $x' = 1 = 1!$.
- Let $n \geq 1$. Assume that $[x^n]^{(n)} = n!$. Then we have

$$[x^{n+1}]^{(n+1)} = [(x^{n+1})']^{(n)} = [(x^n x)']^{(n)} = [(nx^{n-1}x + x^n)']^{(n)} = [(n+1)x^n]^{(n)} = (n+1)[x^n]^{(n)}.$$

By induction hypothesis, $[x^n]^{(n)} = n!$, hence

$$[x^{n+1}]^{(n+1)} = (n+1)[x^n]^{(n)} = (n+1)n! = (n+1)!.$$

Exercise 2.

$$f(x) = \begin{cases} x^6 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Exercise 4.

$$f(x) = \frac{\sin(x^2)}{x}.$$

Exercise 6. The function $f(x) = e^x$ is infinitely many times differentiable on $\mathbb{R}$, and for all $n \geq 1$,

$$f^{(n)}(x) = e^x.$$

Hence, for all $x \in \mathbb{R}$, the Taylor theorem tells us that there exists c between x and 0 such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c).$$

Since $f^{(n)}(0) = 1$, we have

$$e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c).$$

Exercise 7. 1. $f(x) = \frac{1}{1-x}$. The function f is infinitely many times differentiable on $(-\infty, 1)$.

One can prove, by induction, that for all $n \geq 1$, for all $x < 1$,

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

Hence, by Taylor formula, for all $x < 1$, there exists c between 0 and x such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c).$$

The Taylor polynomial of f centered about 0 is, by definition,

$$f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) = 1 + x + x^2 + \cdots + x^n.$$

Exercise 8.

- Since the functions $\cos(x)$ and $1 - \frac{x^2}{2}$ are even, it is enough to prove the inequality for $x \geq 0$. Indeed, assume that the inequality

$$\cos(x) \geq 1 - \frac{x^2}{2}$$

is true for all $x \geq 0$. If so, then for all $x \leq 0$, since $-x \geq 0$, we have

$$\cos(-x) \geq 1 - \frac{(-x)^2}{2}.$$

Since $\cos(-x) = \cos(x)$ we thus have

$$\cos(x) \geq 1 - \frac{x^2}{2}.$$

- Let us thus prove the inequality for $x \geq 0$. Note that $1 - \frac{x^2}{2} \leq -1 \Longleftrightarrow x \geq 2$. Hence, for all $x \geq 2$,

$$\cos(x) \geq -1 \geq 1 - \frac{x^2}{2}.$$

- It remains to prove the inequality for $x \in [0, 2]$. For $x = 0$, the inequality clearly holds. Assume now that $x > 0$. Since $f(x) = \cos(x)$ is infinitely many times differentiable on $\mathbb{R}$, by Taylor formula, there exists $c \in (0, x)$ such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{x^3}{6}f'''(c).$$

Since

$$f(x) = \cos(x), \quad f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f'''(x) = \sin(x),$$

we deduce that

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^3}{6}\sin(c).$$

Since $\sin(x) \geq 0$ for $x \in [0, \pi]$, we deduce that $\sin(c) \geq 0$ (because $c \in [0, 2]$). Hence,

$$\cos(x) \geq 1 - \frac{x^2}{2}.$$