Solution - Homework #4

Exercise 1.

Since $\{B_t\}$ is a Brownian motion, $B_t - B_s$ is independent of B_s, and $B_t - B_s \sim \mathcal{N}(0, t-s)$, and $B_s \sim \mathcal{N}(0, s)$. Without loss of generality, assume $s \le t$. Then,

$$\mathrm{Cov}(B_t, B_s) = \mathbb{E}[B_t B_s] = \mathbb{E}[(B_t - B_s + B_s)B_s] = \mathbb{E}[(B_t - B_s)B_s] + \mathbb{E}[B_s^2]$$
$$= \mathbb{E}[B_t - B_s]\mathbb{E}[B_s] + \mathbb{E}[B_s^2] = s = \min(s,t).$$

Exercise 2. • B_t is $\mathcal{F}_t$ measurable by assumption.
• $B_t \sim \mathcal{N}(0,t)$, hence $\mathbb{E}[|B_t|] < +\infty$.
• Let $s \le t$. Then,

$$\mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s|\mathcal{F}_s] + B_s.$$

Since $\{B_t\}$ is a Brownian motion, $B_t - B_s$ is independent of B_s, and $B_t - B_s \sim \mathcal{N}(0, t-s)$, thus

$$\mathbb{E}[B_t - B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s] = 0.$$

Exercise 3. 1. • Consider, for example, the characteristic function to see that $X_t \sim \mathcal{N}(0,t)$.
• For all $\omega \in \Omega$, the function $t \mapsto \sqrt{t}\, Z(\omega)$ is continuous. Hence, X_t has surely continuous paths.

2. No, since $X_t - X_s$ does not have variance $t - s$.

Exercise 4.

Yes.

Exercise 5.

1. $\widetilde{X}_0 = \widetilde{X}_1$ a.s. is clear, as well as almost sure continuity of the paths. For all $t \ge 0$, $\mathbb{E}[\widetilde{X}_t] = 0$, hence

$$\mathrm{Cov}(\widetilde{X}_t, \widetilde{X}_s) = \mathbb{E}[\widetilde{X}_t\widetilde{X}_s] = \mathbb{E}[B_tB_s] - s\mathbb{E}[B_tB_1] - t\mathbb{E}[B_sB_1] + ts\mathbb{E}[B_1^2] = \min(s,t) - st - st + ts = \min(s,t) - st.$$

Since $\{B_t\}$ is a Gaussian process, $\{\widetilde{X}_t\}$ is also a Gaussian process (take an arbitrary linear combination of $\widetilde{X}_{t_1}, \dots, \widetilde{X}_{t_n}$).

2. **Lemma:** A stochastic process $\{X_t\}$ is a Brownian motion if and only if its paths are almost surely continuous and $\{X_t\}$ is a centered Gaussian process with $\mathrm{Cov}(X_t, X_s) = \min(s,t)$.

We are going to use the above lemma to check that $\{\widetilde{B}_t\}$ is a Brownian motion. The fact that the paths are continuous a.s. is clear. Since Z is independent of $\{X_t\}$ and $\{X_t\}$ is a centered Gaussian process, $\{\widetilde{B}_t\}$ is a centered Gaussian process. Lastly, we easily check that

$$\mathrm{Cov}(\widetilde{B}_t, \widetilde{B}_s) = \min(s,t).$$

3. Use the lemma above again.

Exercise 6.

1. Since $B_1 \sim \mathcal{N}(0,1)$,

$$\mathbb{P}(B_1 \geq 0) = \frac{1}{2}.$$

2.

$$\mathbb{P}(B_2 \geq 0, B_1 \geq 0) = \mathbb{P}(B_2 - B_1 + B_1 \geq 0, B_1 \geq 0).$$

Putting $X = B_1$ and $Y = B_2 - B_1$ we see that

$$\mathbb{P}(B_2 \geq 0, B_1 \geq 0) = \mathbb{P}(Y \geq -X, X \geq 0) = \mathbb{P}((X,Y) \in A),$$

where $A = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq -x\} \subset \mathbb{R}^2$. Since X, Y i.i.d. $\mathcal{N}(0,1)$, we see (draw a picture of A in the plane) that

$$\mathbb{P}((X,Y) \in A) = \frac{3}{8}.$$

3. Same idea.