University of Florida **MAA 4102/5104**

Homework #6 – **Continuity of a function, properties of continuous functions**

PARTIAL SOLUTIONS

Exercise 1. Determine where the given functions are continuous. Explain clearly. (First, find the domain of the given functions.)

1. $f(x) = |x-1|$.

   The domain of the function f is $D = \mathbb{R}$. Fix $a \in \mathbb{R}$ arbitrary, and let us prove that f is continuous at a. Let $\varepsilon > 0$. For all $x \in \mathbb{R}$, we have by the triangle inequality

   $$|f(x) - f(a)| = \big||x-1| - |a-1|\big| \leq |(x-1) - (a-1)| = |x-a|. \tag{1}$$

   So, choose $\delta = \varepsilon$. Hence, if $|x-a| \leq \delta = \varepsilon$, then from (1),

   $$|f(x) - f(a)| \leq |x-a| \leq \varepsilon.$$

   This proved continuity of f at a. Since $a \in \mathbb{R}$ was arbitrary, this shows that f is continuous on $\mathbb{R}$.

2. $f(x) = \frac{|x|}{x}$.

   The domain of f is $D = \mathbb{R} \setminus \{0\}$. Note that if $x < 0$, then $f(x) = -1$, and if $x > 0$, then $f(x) = 1$. Hence, it is easy to check that f is continuous on $\mathbb{R} \setminus \{0\}$.

3. $$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

   The domain of f is $D = [0, +\infty)$. f is not continuous at 0. One way to prove this is to use the sequential equivalent property of continuity.

   Recall that a function f is continuous at a if and only if for all sequence $\{x_n\}$ that converges to a, the sequence $\{f(x_n)\}$ converges to $f(a)$.

   Here, consider $x_n = \frac{1}{n}$, $n \geq 1$. We have $\lim_n x_n = 0$, however $\lim_n f(x_n) = 1 \neq 0 = f(0)$. This shows that f is not continuous at 0

Exercise 2. Prove that if the functions $f, g \colon D \to \mathbb{R}$ are continuous at a, then the given functions are also continuous at a.

1. $|f|$

   Assume that f is continuous at a. Fix $\varepsilon > 0$. Since f is continuous at a, there exists $\delta > 0$ such that for all $x \in D$, if $|x-a| \leq \delta$, then $|f(x) - f(a)| \leq \varepsilon$.

   In particular, for all $x \in D$, if $|x-a| \leq \delta$, then by the triangle inequality

   $$\big||f(x)| - |f(a)|\big| \leq |f(x) - f(a)| \leq \varepsilon.$$

   This proves that $|f|$ is continuous at a (same δ as for f).