Homework #7 – **Uniform Continuity**

PARTIAL SOLUTIONS

Exercise 1. Determine whether or not the given functions are uniformly continuous.

1. $f(x) = \frac{1}{x}$, with $x \in (0, 1]$
   NO
2. $f(x) = x^3$, with $x \in [0, 2)$
   YES
5. $f(x) = \frac{1}{x^2}$, with $x \in [1, +\infty)$
   YES

Exercise 2. If f and g are uniformly continuous on D, prove that:

4. If D is bounded, then f is bounded.

   Take $\varepsilon = 1$. Since f is uniformly continuous on D, there exists $\delta > 0$ such that for all $x, y \in D$, if $|x - y| \leq \delta$, then $|f(x) - f(y)| \leq 1$.

   Since D is bounded, there exists $N \in \mathbb{N}$ such that

   $$D \subset [-N\delta, N\delta].$$

   Assume, without loss of generality, that for all $k \in \{-N, \ldots, N-1\}$,

   $$[k\delta, (k+1)\delta] \cap D \neq \emptyset.$$

   For each $k \in \{-N, \ldots, N-1\}$, choose $x_k \in [k\delta, (k+1)\delta] \cap D$. Now, let $x \in D$ be arbitrary. Then, there exists $k_0 \in \{-N, \ldots, N-1\}$ such that $|x - x_{k_0}| \leq \delta$. (Exercise. Draw a picture to help.)

   Since $|x - x_{k_0}| \leq \delta$, we have $|f(x) - f(x_{k_0})| \leq 1$. We deduce, by the triangular inequality, that

   $$|f(x)| \leq |f(x_{k_0})| + 1 \leq M + 1,$$

   where $M = \sup_{k \in \{-N, \ldots, N-1\}} |f(x_k)| < +\infty$ (it is finite because we are taking the supremum over a finite set).

Exercise 3. Give an example of a function f_1, f_2, f_3, f_4 that satisfies each of the given conditions.

3. f_3 uniformly continuous on $[0, +\infty)$ with $\lim_{x \to +\infty} f_3(x) = -\infty$.

   Consider $f_3(x) = -x$, defined on $\mathbb{R}$.

Exercise 4. Give an example of a function f with a Lipschitz constant $L \geq 1$, such that

1. f has no fixed points.

   Consider $f(x) = x + 1$, defined on $\mathbb{R}$.