Homework #8 – **Derivative of a function**

PARTIAL SOLUTIONS

Exercise 2.

1. $f(x) = c$. Here the domain is $\mathbb{R}$. Let us prove that f is differentiable on $\mathbb{R}$. Let $a \in \mathbb{R}$ be arbitrary. For all $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = 0.$$

Hence,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = 0.$$

We conclude that f is differentiable at a, and $f'(a) = 0$. Since a was arbitrary, we deduce that f is differentiable on $\mathbb{R}$, and that for all $a \in \mathbb{R}$, $f'(a) = 0$. f' is a constant function equal to 0.

2. $f(x) = \sqrt{x}$. Here the domain is $[0, +\infty)$.
At $a = 0$: For all $x > 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{1}{\sqrt{x}}.$$

Since $\lim_{x \to 0} \frac{1}{\sqrt{x}} = +\infty$, we conclude that f is not differentiable at 0.
At $a > 0$. For all $x \neq a$,

$$\frac{f(x) - f(a)}{x - a} = \frac{\sqrt{x} - \sqrt{a}}{x - a} = \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} = \frac{1}{\sqrt{x} + \sqrt{a}}.$$

Hence,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \frac{1}{2\sqrt{a}}.$$

We conclude that f is differentiable at a, and $f'(a) = \frac{1}{2\sqrt{a}}$. Since $a > 0$ was arbitrary, we deduce that f is differentiable on $(0, +\infty)$, and that for all $a > 0$, $f'(a) = \frac{1}{2\sqrt{a}}$.

Exercise 3.

1.
$$f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \qquad \text{at } x = 0$$

At 0: For all $x \neq 0$,

$$\frac{f(x) - f(0)}{x - 0} = \sin\left(\frac{1}{x}\right).$$

Since $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist (justification below), we conclude that f is not differentiable at 0.

Justification: $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist:

Proof: Construct 2 sequences $\{x_n\}$ and $\{y_n\}$ converging to 0 and such that $\{\sin\left(\frac{1}{x_n}\right)\}$ and $\{\sin\left(\frac{1}{y_n}\right)\}$ have different limits. For example, take

$$x_n = \frac{1}{2\pi n}, \qquad y_n = \frac{1}{\frac{\pi}{2} + 2\pi n}.$$

Thus, $\forall n \geq 1$, $\sin\left(\frac{1}{x_n}\right) = \sin(2\pi n) = 0$ and $\sin\left(\frac{1}{y_n}\right) = \sin(\frac{\pi}{2} + 2\pi n) = 1$. Hence,

$$\lim_{n\to+\infty} \sin\left(\frac{1}{x_n}\right) = 0, \qquad \lim_{n\to+\infty} \sin\left(\frac{1}{y_n}\right) = 1.$$

This proves that $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Comment: On $\mathbb{R} \setminus \{0\}$, the function f is differentiable since it is a composition/product of differentiable functions on their respective domain ($\sin(x)$ on $\mathbb{R}$, x on $\mathbb{R}$, $\frac{1}{x}$ on $\mathbb{R} \setminus \{0\}$).

Exercise 4.

2.

$$f(x) = \begin{cases} |x| & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

This function f is defined on $\mathbb{R}$, continuous at 0 only, and is nowhere differentiable (justification below).

Proof: • Continuity at 0: Let $x \in \mathbb{R}$. If $x \in \mathbb{Q}$, then $|f(x)| = |x|$, and if $x \in \mathbb{R} \setminus \mathbb{Q}$, then $|f(x)| = 0$. We just proved that, in any case, if $x \in \mathbb{R}$, then $|f(x)| \leq |x|$. We conclude, by squeeze theorem, that $\lim_{x\to 0} f(x) = 0 = f(0)$. Hence, f is continuous at 0.

• f is not continuous on $\mathbb{R} \setminus \{0\}$: Let $a \neq 0$, and take any sequence of rational $\{x_n\}$ and irrational $\{y_n\}$ converging to a. For these 2 sequences, $\{f(x_n)\}$ and $\{f(y_n)\}$ have different limit. Hence, f is not continuous at a. Since a is arbitrary in $\mathbb{R} \setminus \{0\}$, the function f is not continuous on $\mathbb{R} \setminus \{0\}$.

• Since differentiability implies continuity, we deduce that f is not differentiable on $\mathbb{R} \setminus \{0\}$ (because it is not continuous on $\mathbb{R} \setminus \{0\}$).

• f is not differentiable at 0 because

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist (take a sequence of positive rational numbers, and positive irrational numbers converging to 0).