University of Florida **MAA 4102/5104**

Homework #9 – **Properties of differentiable function, Mean value theorems**

PARTIAL SOLUTIONS

Exercise 10.

The function $f(x) = x^3 - x$ is continuous on $[0,1]$, differentiable on $(0,1)$, and satisfies $f(0) = f(1) = 0$. Hence by the Rolle theorem, there exists $c \in (0,1)$ such that

$$f'(c) = 0.$$

Since $f'(x) = 3x^2 - 1$, we conclude that there exists $c \in (0,1)$ such that

$$3c^2 - 1 = 0.$$

Exercise 12.

1. Let $x, y \in (a,b)$, with $x \neq y$. Since f is differentiable on (a,b), f is continuous on $[x,y]$ and differentiable on (x,y). Hence, by the mean value theorem, there exists c between x and y such that

$$\frac{f(y) - f(x)}{y - x} = f'(c).$$

By assumption, f' is bounded on (a,b), which means that

$$\exists M > 0, \forall c \in (a,b), |f'(c)| \leq M.$$

We deduce that

$$\left| \frac{f(y) - f(x)}{y - x} \right| = |f'(c)| \leq M.$$

We conclude that for all $x, y \in (a,b)$,

$$|f(y) - f(x)| \leq M|y - x|.$$

In particular, f is a Lipschitz function on (a,b), and thus f is uniformly continuous on (a,b).

2. Consider $f(x) = x\sin(\frac{1}{x})$ on $(0,1)$.

Exercise 13.

1. Let $r \geq 0$. If $x = 0$, then the inequality clearly holds. Assume now that $x > 0$, and consider the function $f(t) = (1+t)^r$ on $[0, +\infty)$. Since f is continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$, by the mean value theorem, there exists $c \in (0,x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(c).$$

Since $c > 0$, $(1+c)^{r-1} \geq 1$, and thus

$$f'(c) = r(1+c)^{r-1} \geq r.$$

We deduce that
$$\frac{(1+x)^r - 1}{x} = r(1+c)^{r-1} \geq r.$$
Multiplying both side by $x > 0$, we have
$$(1+x)^r \geq rx + 1,$$
which is the desired inequality.

2. Same argument, applied to the function $f(x) = e^x$:

- If $x = 0$, then the inequality clearly holds.
- If $x < 0$, then, Since f is continuous and differentiable on $\mathbb{R}$, by the mean value theorem, there exists $c \in (x, 0)$ such that
$$\frac{f(x) - f(0)}{x - 0} = f'(c) = e^c.$$
Since $c < 0$, $e^c \leq 1$, and thus
$$\frac{e^x - 1}{x} \leq 1.$$
Multiplying both side by $x < 0$ (hence reversing the inequality), we have
$$e^x \geq 1 + x.$$

- If $x > 0$, the same argument applies.