Lecture 17: Section 2.6
Rational Functions

Holes and Vertical Asymptotes

The graph of a rational function \( f(x) = \frac{p(x)}{q(x)} \) has a hole at \( x = a \) if

1. both \( p(x) \) and \( q(x) \) have the common factor \( (x - a) \), and

2. the simplified denominator DOES NOT have the factor \( (x - a) \).

NOTE: To find the \( y \)-coordinate of the hole in the graph, plug \( x = a \) into the simplified expression of \( f(x) \).

\textbf{ex.} Find all vertical asymptotes and holes of the function:

1) \( f(x) = \frac{x + 1}{x^2 - x - 2} \)
2) \[ g(x) = \frac{2x^3(x + 1)(x - 3)}{x(x + 1)^2(x - 2)(x - 3)} \]
Horizontal Asymptotes

Def. The line \( y = L \) is a horizontal asymptote (HA) of the graph \( y = f(x) \) if

To Find Horizontal Asymptotes:

Let \( f(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + \cdots + a_0}{b_m x^m + \cdots + b_0} \).

1. If \( n < m \), \( f(x) \) is a proper rational function. The graph of \( f \) has a horizontal asymptote

2. If \( n = m \), the graph of \( f(x) \) has a horizontal asymptote

3. If \( n > m \), the graph of \( f(x) \) has
ex. Find all horizontal asymptotes of the following:

1) \( f(x) = \frac{x}{x^2 + 1} \)

2) \( f(x) = \frac{x^2 - 4}{2x + 1} \)

3) \( f(x) = \frac{3x^2 + 9}{3 - 2x^2} \)

NOTE: A graph can cross its horizontal asymptote!
To Graph a Rational Function:

1. Factor numerator and denominator.
2. Find the domain.
3. Reduce the fraction to lowest terms.
4. Find each vertical asymptote.
5. Find the coordinates of any holes in the function.
6. Find the horizontal asymptote if there is one. Does the graph of the function cross its horizontal asymptote? If so, at what point?
7. Find any $x$-intercept and $y$-intercept.
8. Check for symmetry.
9. Use the zeros of both numerator and denominator of the reduced rational function to divide the number line into test intervals. Choose a point in each interval to determine where the graph lies above or below the $x$-axis.
10. Sketch the asymptotes and plot the points found above; graph the function.
ex. Sketch the graph of $f(x) = \frac{2x^2 + 7x - 4}{x^2 + x - 2}$. 
ex. Sketch the graph of \( f(x) = \frac{x^2 - 1}{x^2 - 4} \).
**ex.** Sketch the graph of \( f(x) = \frac{2x^2 - 4x}{x^2 + x - 6}. \)
Applications

ex. The rabbit population on Mr. Jenkins’ farm follows the model

\[ p(t) = \frac{3000t}{t + 1} \]

where \( t \geq 0 \) is the time (in months) since the beginning of the year.

1) Draw a graph of the rabbit population.

2) What eventually happens to the rabbit population?