THE LIMIT OF A FUNCTION, \( f(x) \):

The limit of \( f(x) \) as \( x \) approaches \( a \) is denoted \( \lim_{x \to a} f(x) \). This limit determines the "value" of \( f(x) \) in a neighborhood around \( x \).

ONE-SIDED LIMIT:

* \( \lim_{x \to a^+} f(x) \) is the limit of \( f(x) \) as \( x \) approaches \( a \) from the right

* \( \lim_{x \to a^-} f(x) \) is the limit of \( f(x) \) as \( x \) approaches \( a \) from the left

* If \( \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \), then \( \lim_{x \to a} f(x) \) exists

* If \( \lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \), then \( \lim_{x \to a} f(x) \) does not exist

HOW TO FIND LIMITS (WHEN \( x \) IS NOT APPROACHING \( \pm \infty \)):

* METHOD ONE: GRAPH THE FUNCTION AND FIND THE LIMIT BY ANALYZING THE GRAPH

* METHOD TWO: FIND LIMIT ALGEBRAICALLY

**STEP ONE:** Plug in the value that \( x \) is approaching. If you get an answer, then this is your limit

**STEP TWO:** Simplify \( f(x) \) and then plug in the value that \( x \) is approaching to the simplified \( f(x) \), if you get an answer, then this is your limit

**STEP THREE:** When you plug in the number that \( x \) is approaching, your answer will look like: \( \frac{\text{NUMBER}}{0} \). If
DISCUSS HOW TO SOLVE BELOW.

INFINITE LIMITS:

CONTINUING FROM STEP THREE ABOVE, IF YOU GET THE FORM
"NUMBER", THEN PLUG IN A NUMBER CLOSE TO THE NUMBER
0

BEING APPROACHED. (I TYPICALLY ALWAYS PLUG IN A NUMBER
WITHIN 0.1). THEN, ANALYZE THE SIGN.

EXAMPLE: \( \lim_{{x \to 2^+}} \frac{x}{x-2} = \frac{2}{2-2} = \frac{2}{0} \) "NUMBER"

SINCE \( x \to 2^+ \) CHOOSE A NUMBER TO THE RIGHT OF 2, SAY
2.1. \( x = 2.1 \) = \( \frac{2}{2.1-2} \) \( \frac{2}{0.1} \) \( \frac{2}{0} \) \( \frac{2}{0} \)

\[ \lim_{{x \to 2^+}} \frac{x}{x-2} = +\infty \]

EXAMPLE: \( \lim_{{x \to 2^-}} \frac{x}{x-2} = \frac{2}{2-2} = \frac{2}{0} \) "NUMBER"

SINCE \( x \to 2^- \) CHOOSE A NUMBER TO THE LEFT OF 2, SAY
1.9. \( x = 1.9 \) = \( \frac{2}{1.9-2} \) \( \frac{2}{-0.1} \) \( \frac{2}{0} \) \( \frac{2}{0} \)

\[ \lim_{{x \to 2^-}} \frac{x}{x-2} = -\infty \]

EXAMPLE: \( \lim_{{x \to 2}} \frac{x}{x-2} \) DOES NOT EXIST BECAUSE
\[ \lim_{{x \to 2^+}} = +\infty \] \( \lim_{{x \to 2^-}} = -\infty \)

\[ 80 \lim_{{x \to 2^+}} \neq \lim_{{x \to 2^-}} \]
LIMITS TO REMEMBER:

\[ \lim_{x \to 0^+} \frac{1}{x} = +\infty \]

\[ \lim_{x \to 0} \ln(x) = -\infty \]

LIMIT LAWS:

\[ \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) \]

\[ \lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x) \]

\[ \lim_{x \to a} \left[ \frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \]

\[ \lim_{x \to a} cf(x) = c \lim_{x \to a} f(x) \quad (c \text{ is a constant}) \]

\[ \lim_{x \to a} f(g(x)) = f \left( \lim_{x \to a} g(x) \right) \]

\[ \lim_{x \to a} [f(x)]^n = \left( \lim_{x \to a} f(x) \right)^n \]

SQUEEZE THEOREM:

If \( f(x) \leq g(x) \leq h(x) \) and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \), then also \( \lim_{x \to a} g(x) = L \).
HOW TO SOLVE:

1. START WITH A TRUE STATEMENT
2. MANIPULATE INEQUALITY UNTIL THE MIDDLE LOOKS LIKE YOUR ORIGINAL FUNCTION
3. TAKE THE LIMIT OF EACH PART OF THE INEQUALITY

EXAMPLE: \( \lim_{x \to 0} x^3 \cos(x) \)

1. \(-1 \leq \cos(x) \leq 1\) (BECUSE THE RANGE OF \( \cos(x) \) IS \([-1,1]\).)
2. \(-x^3 \leq x^3 \cos(x) \leq x^3\) (MULTIPLY BY \( x^3 \) TO GET ORIGINAL FUNCTION)
3. \( \lim_{x \to 0} -x^3 \leq \lim_{x \to 0} x^3 \cos(x) \leq \lim_{x \to 0} x^3\)  

\( 0 \leq \lim_{x \to 0} x^3 \cos(x) \leq 0\)

\( \Rightarrow \lim_{x \to 0} x^3 \cos(x) = 0\)

ABSOLUTE VALUE LIMITS: USE DEFINITION OF ABSOLUTE VALUE FUNCTION TO DETERMINE IF POSITIVE OR NEGATIVE.

EXAMPLE: \( \lim_{x \to 3^+} |x-3| \)

LOOK AT \( |x-3| \): WHEN \( x \to 3^+ \), \( x > 3 \) (BECAUSE \( x \) IS APPROACHING 3 FROM THE RIGHT AND 3 TO THE RIGHT OF 3 ARE BIGGER THAN 3), SO \( |x-3| = x-3 \)

\( \Rightarrow \lim_{x \to 3^+} |x-3| = \lim_{x \to 3^+} x - 3 = 1 \)
Example: \( \lim_{x \to 3^-} \frac{|x-3|}{x^2 - 9} \)

Look at \( |x-3| \): When \( x \to 3^- \), \( x < 3 \) (because \( x \) is approaching \( 3 \) from the left, and \( x \)'s to the right of \( 3 \) are smaller than \( 3 \)), so \( |x-3| = -(x-3) \)

\[
\Rightarrow \lim_{x \to 3^-} \frac{|x-3|}{x^2 - 9} = \lim_{x \to 3^-} \frac{-(x-3)}{(x+3)(x-3)} = \lim_{x \to 3^-} \frac{-1}{x+3} = -\frac{1}{6}
\]

Note: \( |x| = \begin{cases} \frac{x}{x} & x > 0 \\ -x & x < 0 \end{cases} \)