6 Polynomial Functions

6.1 End and Zero Behavior

Note 1. A polynomial of degree 2 or more has a graph with no sharp turns or cusps.

Note 2. The domain of a polynomial function is ________________.

Definition

The values of $x$ for which $f(x) = 0$ are called the ________________ or $x$-intercepts of $f$.

Note 3. If a polynomial can be factored, we can set each factor equal to zero to find the $x$-intercepts (or zeros) of the function. Recall that the $x$-intercepts of a function are where $f(x) = 0$, or $y = 0$. The $y$-intercepts are where $x = 0$.

How to Find the $x$-Intercepts of a Polynomial Function, $f$, by Factoring

1. Set ________________.

2. If the polynomial function is not in factored form, then factor the polynomial.
3. Set each factor equal to _____ to find the x-intercepts.

**Example 1.** Find the x and y-intercepts of:

\[ g(x) = (x - 2)^2(2x + 3) \]

**Note 4.** The graphs of polynomials behave differently at various x-intercepts. Sometimes, a graph will ______________ the horizontal x-axis at the x-intercepts, and other times the graph will ______________ or bounce off the horizontal x-axis at the x-intercepts.

**Definition**

The number of times a given factor appears in the factored form of a polynomial is called the ______________.

**Example 2.** From the above example, \( g(x) = (x - 2)^2(2x + 3) \), the factor associated to the zero at \( x = 2 \) has multiplicity ___. This zero has even multiplicity. The factor associated to the zero at \( x = -\frac{3}{2} \) has multiplicity ___. This zero has odd multiplicity.

**Graphical Behavior of Polynomials at x-Intercepts (Zeros)**

If a polynomial contains a factor in the form \( (x - h)^p \), the behavior near the x-intercept \( h \) is determined by the power \( p \). We say that \( x = h \) is a zero of ______________ \( p \). The graph of a polynomial function will touch the x-axis at zeros with ______________ multiplicities. The graph of a polynomial function will cross the x-axis at zeros with ______________ multiplicities. The sum of the multiplicities is the ______________ of the polynomial function.
**Example 3.** The graphs below exemplify the behavior of polynomials at their zeros with different multiplicities:

Note 5. The graph of a polynomial function of the form

\[ f(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + a_n x^n \]

will either ______ or _______ as \( x \) increases without bound and will either ______ or _______ as \( x \) decreases without bound. This is called the ______ _______ of a function.
Example 4. The chart below illustrates the end behavior of a polynomial function:

<table>
<thead>
<tr>
<th></th>
<th>Even Degree</th>
<th>Odd Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive Leading</td>
<td>$a_n &gt; 0$</td>
<td>Positive Leading Coefficient</td>
</tr>
<tr>
<td>Coefficient</td>
<td></td>
<td>$a_n &gt; 0$</td>
</tr>
<tr>
<td>End Behavior:</td>
<td>$x \to \infty, f(x) \to \infty$</td>
<td>End Behavior:</td>
</tr>
<tr>
<td></td>
<td>$x \to -\infty, f(x) \to \infty$</td>
<td>$x \to \infty, f(x) \to -\infty$</td>
</tr>
<tr>
<td>Negative Leading</td>
<td>$a_n &lt; 0$</td>
<td>Negative Leading Coefficient</td>
</tr>
<tr>
<td>Coefficient</td>
<td></td>
<td>$a_n &lt; 0$</td>
</tr>
<tr>
<td>End Behavior:</td>
<td>$x \to \infty, f(x) \to -\infty$</td>
<td>End Behavior:</td>
</tr>
<tr>
<td></td>
<td>$x \to -\infty, f(x) \to -\infty$</td>
<td>$x \to -\infty, f(x) \to \infty$</td>
</tr>
</tbody>
</table>
Example 5. Choose the end behavior of the polynomial function:

\[ f(x) = -(x + 7)^6 (x + 5)^4 (x - 5)^3 (x - 7)^3 \]
Example 6. Choose the option below that describes the behavior at $x = -3$ of the polynomial:

$$f(x) = (x + 6)(x + 3)^4(x - 3)^3(x - 6)$$
Example 7. Choose the option below that describes the behavior at $x = -9$ of the polynomial:

$$f(x) = (x + 9)^3(x + 3)^5(x - 3)^3(x - 9)^2$$
Example 8. Choose the end behavior of the polynomial function:

\[ f(x) = (x + 9)(x + 4)^6(x - 4)^3(x - 9) \]
6.2 Graphing Polynomials

**Definition**

A __________ _________ of the graph of a polynomial function is the point where a function changes from rising to falling or from falling to rising. A polynomial of degree \( n \) will have at most ___________ turning points.

**How to Determine the Zeros and Multiplicities of a Polynomial of Degree \( n \) Given its Graph**

1. If the graph crosses the \( x \)-axis at the intercept, it is a zero with __________ _________.
2. If the graph touches the \( x \)-axis and bounces off the axis, it is a zero with __________ _________.
3. The sum of the multiplicities is _____.

**Definition**

If a polynomial of lowest degree \( p \) has \( x \)-intercepts at \( x = x_1, x_2, \ldots, x_n \), then the polynomial can be written in factored form:

_______________________________.
Note 6. In the factored form of a polynomial, the powers on each factor can be determined by the behavior of the graph at the corresponding ________________, and the stretch factor $a$ can be determined given a value of the function other than the _____ - ________________.

**How to Determine a Polynomial Function Given its Graph**

1. Identify the _____ - _________________ to determine the factors of the polynomial.

2. Examine the behavior of the graph at $x$-intercepts to determine the _________________ of each factor.

3. Find the polynomial of least degree containing all the factors found in step 2.

4. Use any other point on the graph (typically the $y$-intercept) to determine the stretch factor (or, you can analyze the end behavior of the graph to determine the stretch factor).
Example 9. Write an equation of the function graphed below:
Example 10. Write an equation of the function graphed below:
How to Sketch the Graph of a Polynomial Function

1. Find the \( x \)-intercepts (zeros).

2. Find the \( y \)-intercepts.

3. Check for symmetry. If the function is an even function, then its graph is symmetric about the \(-\) - \(-\)-______________ (that is, \( f(-x) = f(x) \)). If the function is an odd function, then its graph is symmetric about the \(-\) - \(-\)-______________ (that is, \( f(-x) = -f(x) \)).

4. Determine the behavior of the polynomial at the zeros using their ________________.

5. Determine the ________________ ________________.

6. Sketch a graph.

7. Check that the number of ________________ ________________ does not exceed one less than the degree of the polynomial.
The Factor Theorem

\( k \) is a zero of \( f(x) \) if and only if \( \quad \) is a factor of \( f(x) \).

**Note 7.** The following statements are equivalent:

**Note 8.** If we are given the zeros of a polynomial, we can use the \( \quad \) \( \quad \) to construct the lowest-degree polynomial.

**Example 11.** Construct the lowest-degree polynomial given the zeros below:

\[ 3, -3, -4 \]
Example 12. Construct the lowest-degree polynomial given the zeros below:

\[- \frac{4}{3}, -\frac{3}{2}, -3\]

Fundamental Theorem of Algebra

If \( f(x) \) is a polynomial of degree \( n > 0 \), then \( f(x) \) has at least one \( \underline{\text{____________}} \) \( \underline{\text{____________}} \). In fact, if \( f(x) \) is a polynomial of degree \( n > 0 \) and \( a \) is a nonzero real number, then \( f(x) \) has exactly \( n \) \( \underline{\text{____________}} \) \( \underline{\text{____________}} \):

\[ \underline{\text{________________________________________}} \]

where \( c_1, c_2, ..., c_n \) are complex numbers. That is, \( f(x) \) has \( \underline{\text{____________}} \) if we allow for multiplicities.

Note 9. This does NOT mean that every polynomial has an imaginary zero. Real numbers are a subset of the complex numbers, but complex numbers are not a subset of the real numbers.
The Linear Factorization Theorem

If \( f \) is a polynomial function of degree \( n \), then \( f \) has \( n \) ____________, and each factor is of the form ____________, where \( c \) is a complex number. That is, a polynomial function has the same number of linear factors as its degree.

Complex Conjugate Theorem

Suppose \( f \) is a polynomial function with real coefficients. If \( f \) has a complex zero of the form \( a + bi \), then the ________________ ________________ of the zero, \( a - bi \), is also a zero.

A Closer Look at the Zeros of a Polynomial Function

Recall the quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

Case 1: \( b^2 - 4ac \) is positive and not a perfect square:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

Case 2: \( b^2 - 4ac \) is negative:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]
Example 13. Construct the lowest-degree polynomial given the zeros below:

\[ \sqrt{2}, \frac{1}{3} \]

Example 14. Construct the lowest-degree polynomial given the zeros below:

\[ 4 + 3i, -\frac{2}{5} \]