Module 6 Lecture Notes
MAC1105
Summer B 2019

6 Polynomial Functions

6.1 End and Zero Behavior

Note 1. A polynomial of degree 2 or more has a graph with no sharp turns or cusps.

Note 2. The domain of a polynomial function is $(-\infty, \infty)$.

Definition
The values of $x$ for which $f(x) = 0$ are called the zeros or $x$-intercepts of $f$.

Note 3. If a polynomial can be factored, we can set each factor equal to zero to find the $x$-intercepts (or zeros) of the function. Recall that the $x$-intercepts of a function are where $f(x) = 0$, or $y = 0$. The $y$-intercepts are where $x = 0$.

How to Find the x-Intercepts of a Polynomial Function, $f$, by Factoring

1. Set $f(x) = 0$.

2. If the polynomial function is not in factored form, then factor the polynomial.
3. Set each factor equal to \(0\) to find the \(x\)-intercepts.

**Example 1.** Find the \(x\) and \(y\)-intercepts of:

\[ g(x) = (x - 2)^2(2x + 3) \]

\(x\)-intercepts: \[ \begin{align*}
  x - 2 &= 0 & \Rightarrow & & x = 2 \\
  2x + 3 &= 0 & \Rightarrow & & x = -\frac{3}{2} \\
\end{align*} \]

\(x\)-intercepts are \((2,0)\) and \((-\frac{3}{2},0)\).

\(y\)-intercept: \[ g(0) = (0 - 2)^2(2(0) + 3) = (-2)^2(3) = 12 \]

Note 4. The graphs of polynomials behave differently at various \(x\)-intercepts. Sometimes, a graph will **cross** the horizontal \(x\)-axis at the \(x\)-intercepts, and other times the graph will **touch** or bounce off the horizontal \(x\)-axis at the \(x\)-intercepts.

**Definition**

The number of times a given factor appears in the factored form of a polynomial is called the **multiplicity**.

**Example 2.** From the above example, \(g(x) = (x - 2)^2(2x + 3)\), the factor associated to the zero at \(x = 2\) has multiplicity 2. This zero has even multiplicity. The factor associated to the zero at \(x = -\frac{3}{2}\) has multiplicity \(\frac{1}{2}\). This zero has odd multiplicity.

**Graphical Behavior of Polynomials at \(x\)-Intercepts (Zeros)**

If a polynomial contains a factor in the form \((x - h)^p\), the behavior near the \(x\)-intercept \(h\) is determined by the power \(p\). We say that \(x = h\) is a zero of **multiplicity** \(p\). The graph of a polynomial function will touch the \(x\)-axis at zeros with **even** multiplicities. The graph of a polynomial function will cross the \(x\)-axis at zeros with **odd** multiplicities. The sum of the multiplicities is the **degree** of the polynomial function.
Example 3. The graphs below exemplify the behavior of polynomials at their zeros with different multiplicities:

Note 5. The graph of a polynomial function of the form $f(x) = a_0 + a_1x + \ldots + a_{n-1}x^{n-1} + a_nx^n$

will either rise or fall as $x$ increases without bound and will either rise or fall as $x$ decreases without bound. This is called the end behavior of a function.
Example 4. The chart below illustrates the end behavior of a polynomial function:

<table>
<thead>
<tr>
<th>Even Degree</th>
<th>Odd Degree</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Positive Leading Coefficient</strong>&lt;br&gt;$a_n &gt; 0$</td>
<td><strong>Positive Leading Coefficient</strong>&lt;br&gt;$a_n &gt; 0$</td>
</tr>
<tr>
<td><img src="chart1.png" alt="Graph" /></td>
<td><img src="chart2.png" alt="Graph" /></td>
</tr>
<tr>
<td><strong>End Behavior:</strong>&lt;br&gt;$x \to \infty, f(x) \to \infty$&lt;br&gt;$x \to -\infty, f(x) \to \infty$</td>
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</tr>
<tr>
<td><img src="chart3.png" alt="Graph" /></td>
<td><img src="chart4.png" alt="Graph" /></td>
</tr>
<tr>
<td><strong>End Behavior:</strong>&lt;br&gt;$x \to \infty, f(x) \to -\infty$&lt;br&gt;$x \to -\infty, f(x) \to -\infty$</td>
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</tr>
</tbody>
</table>
Example 5. Choose the end behavior of the polynomial function:

\[ f(x) = -(x + 7)^6(x + 5)^4(x - 5)^3(x - 7)^3 \]

- Negative Leading Coefficient!

Degree of polynomial is the sum of the multiplicities: \(6 + 4 + 3 + 3 = 16\)

\(\Rightarrow\) Even degree, negative leading coefficient:
Example 6. Choose the option below that describes the behavior at $x = -3$ of the polynomial:

$$f(x) = (x + 6)(x + 3)^4(x - 3)^3(x - 6)$$

**Zeros:**
1. $x = -6$, odd multiplicity
2. $x = -3$, even multiplicity
3. $x = 3$, odd multiplicity
4. $x = 6$, odd multiplicity

- Positive leading coefficient!
- Degree of $f(x)$ is the sum of the multiplicities:

$$1 + 4 + 3 + 1 = 9$$

$\Rightarrow$ odd degree and positive leading coefficient:

- Plot zeros and sketch graph:
Example 7. Choose the option below that describes the behavior at \( x = -9 \) of the polynomial:

\[
f(x) = (x + 9)^3(x + 3)^5(x - 3)^3(x - 9)^2
\]

Zeros:
1. \( x = -9 \), odd multiplicity
2. \( x = -3 \), odd multiplicity
3. \( x = 3 \), odd multiplicity
4. \( x = q \), even multiplicity

★ Positive Leading Coefficient!
★ Degree of \( f(x) \) is the sum of the multiplicities:

\[
3 + 5 + 3 + 2 = 13
\]

\( \Rightarrow \) Odd degree and positive leading coefficient:

\[\]

Plot zeros and sketch graph:
Example 8. Choose the end behavior of the polynomial function:

\[ f(x) = (x + 9)(x + 4)^6(x - 4)^3(x - 9) \]

\[ f(x) = \text{Positive leading coefficient!} \]

\[ \text{Degree of } f(x) \text{ is the sum of the multiplicities:} \]
\[ 1 + 6 + 3 + 1 = 11 \]

\[ \Rightarrow \text{Odd degree and positive leading coefficient!} \]
6.2 Graphing Polynomials

Definition

A **Turning Point** of the graph of a polynomial function is the point where a function changes from rising to falling or from falling to rising. A polynomial of degree $n$ will have at most $n - 1$ turning points.

How to Determine the Zeros and Multiplicities of a Polynomial of Degree $n$ Given its Graph

1. If the graph crosses the $x$-axis at the intercept, it is a zero with **odd** multiplicity.
2. If the graph touches the $x$-axis and bounces off the axis, it is a zero with **even** multiplicity.
3. The sum of the multiplicities is $n$.

Definition

If a polynomial of lowest degree $p$ has $x$-intercepts at $x = x_1, x_2, ..., x_n$, then the polynomial can be written in factored form:

$$f(x) = (x-x_1)^{p_1} (x-x_2)^{p_2} \cdots (x-x_n)^{p_n}$$

where $p = p_1 + p_2 + \cdots + p_n$.
Note 6. In the factored form of a polynomial, the powers on each factor can be determined by the behavior of the graph at the corresponding **ZEROS**, and the stretch factor **a** can be determined given a value of the function other than the **X-INTERCEPTS**.

### How to Determine a Polynomial Function Given its Graph

1. Identify the **X-INTERCEPTS** to determine the factors of the polynomial.

2. Examine the behavior of the graph at x-intercepts to determine the **MULTIPLICITY** of each factor.

3. Find the polynomial of least degree containing all the factors found in step 2.

4. Use any other point on the graph (typically the y-intercept) to determine the stretch factor (or, you can analyze the end behavior of the graph to determine the stretch factor).

\[ \star \text{ IF } x=c \text{ IS A ZERO, THEN } (x-c) \text{ IS A FACTOR OF THE POLYNOMIAL.} \]

\[ f(x) = a(x-c_1)^{m_1}(x-c_2)^{m_2} \ldots (x-c_{m_n})^{m_n} \]

- \(c_1, c_2, \ldots, c_{m_n}\) ARE THE ZEROS AND X-INTERCEPTS
- \(m_1, \ldots, m_n\) ARE THE MULTIPLEILITIES
- \(m_1 + m_2 + \ldots + m_n = m\) IS THE DEGREE OF \(f(x)\)
- \(a\) IS THE LEADING COEFFICIENT
Example 9. Write an equation of the function graphed below:

**X-INTERCEPTS:**

1. $x = 0$, crosses x-axis $\Rightarrow$ odd multiplicity

2. $x = 1$, crosses x-axis $\Rightarrow$ odd multiplicity

3. $x = 2$, crosses x-axis $\Rightarrow$ odd multiplicity

$$\Rightarrow f(x) = q(x - 0)(x - 1)(x - 2)$$
$$= q(x - 1)(x - 2)$$

**End Behavior:**

"Degree" 3
$\Rightarrow$ odd degree and

$\Rightarrow$ positive leading coefficient

$\Rightarrow q = 1$
Example 10. Write an equation of the function graphed below:

**X-INTERCEPTS:**

1. \( x = -1 \), crosses \( x \)-axis \( \Rightarrow \) odd multiplicity
   \((x - (-1)) = (x + 1) \) is a factor with multiplicity 1

2. \( x = 0 \), crosses \( x \)-axis \( \Rightarrow \) odd multiplicity
   \((x - 0) = x \) is a factor with multiplicity 1

3. \( x = 2 \), crosses \( x \)-axis \( \Rightarrow \) odd multiplicity
   \((x - 2) \) is a factor with multiplicity 1

**End Behavior:**

"Degree": 3

Odd degree and end behavior is

\[ f(x) = -(x+1)x(x-2) \]

\[ f(x) = -x(x+1)(x-2) \]
How to Sketch the Graph of a Polynomial Function

1. Find the $x$-intercepts (zeros).

2. Find the $y$-intercepts.

3. Check for symmetry. If the function is an even function, then its graph is symmetric about the $y$-axis (that is, $f(-x) = f(x)$). If the function is an odd function, then its graph is symmetric about the $x$-axis (that is, $f(-x) = -f(x)$).

4. Determine the behavior of the polynomial at the zeros using their multiplicities.

5. Determine the leading coefficient.

6. Sketch a graph.

7. Check that the number of turning points does not exceed one less than the degree of the polynomial.
6.3 Lowest Degree Polynomial

The Factor Theorem

\( k \) is a zero of \( f(x) \) if and only if \( (x - k) \) is a factor of \( f(x) \).

Note 7. The following statements are equivalent:

1. \( x = a \) is a zero of the function \( f \)
2. \( x = a \) is a solution of the equation \( f(x) = 0 \)
3. \( (x-a) \) is a factor of the polynomial \( f(x) \)
4. \( (a,0) \) is an \( x \)-intercept of the graph of \( f(x) \)

Note 8. If we are given the zeros of a polynomial, we can use the **Factor Theorem** to construct the lowest-degree polynomial.

Example 11. Construct the lowest-degree polynomial given the zeros below:

\( 3, -3, -4 \)

**Factors:**

1. \( (x-3) \)
2. \( (x-(-3)) = (x+3) \)
3. \( (x-(-4)) = (x+4) \)

\[
\begin{align*}
\Rightarrow f(x) &= (x-3)(x+3)(x+4) \\
f(x) &= (x^2 + 3x - 3x - 9)(x+4) \\
f(x) &= (x^2 - 9)(x+4) \\
f(x) &= x^3 + 4x^2 - 9x - 36
\end{align*}
\]
Example 12. Construct the lowest-degree polynomial given the zeros below:

\[ \frac{-4}{3}, -\frac{3}{2}, -3 \]

**FACTORS:**

1. \( (x - (-\frac{4}{3})) = x + \frac{4}{3} \)  \( \rightarrow \) **GET RID OF FRACTIONS:** \( (x + \frac{4}{3} = 0) \rightarrow 3x + 4 = 0 \)

2. \( (x - (-\frac{3}{2})) = x + \frac{3}{2} \)  \( \rightarrow \) **GET RID OF FRACTIONS:** \( (x + \frac{3}{2} = 0) \rightarrow 2x + 3 = 0 \)

3. \( (x - (-3)) = x + 3 \)  \( \rightarrow \) **GET RID OF FRACTIONS:** \( f(x) = (x + 3)(3x + 4)(2x + 3) \)

**Fundamental Theorem of Algebra**

If \( f(x) \) is a polynomial of degree \( n > 0 \), then \( f(x) \) has at least one **complex zero**. In fact, if \( f(x) \) is a polynomial of degree \( n > 0 \) and \( a \) is a nonzero real number, then \( f(x) \) has exactly \( n \) **linear factors**:

\[ f(x) = a(x - c_1)(x - c_2) \ldots (x - c_n) \]

where \( c_1, c_2, \ldots, c_n \) are complex numbers. That is, \( f(x) \) has \( n \) **roots** if we allow for multiplicities.

**Note 9.** This does NOT mean that every polynomial has an imaginary zero. Real numbers are a subset of the complex numbers, but complex numbers are not a subset of the real numbers.
The Linear Factorization Theorem

If \( f \) is a polynomial function of degree \( n \), then \( f \) has \( n \) factors, and each factor is of the form \((x - c)\), where \( c \) is a complex number. That is, a polynomial function has the same number of linear factors as its degree.

Complex Conjugate Theorem

Suppose \( f \) is a polynomial function with real coefficients. If \( f \) has a complex zero of the form \( a + bi \), then the complex conjugate of the zero, \( a - bi \), is also a zero.

A Closer Look at the Zeros of a Polynomial Function

Recall the quadratic formula:

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

Case 1: \( b^2 - 4ac \) is positive and not a perfect square:

\[
x = \frac{\text{INTEGER}}{\text{INTEGER}} \pm \frac{\text{IRRATIONAL}}{\text{INTEGER}}
\]

\[
x = \frac{\text{RATIONAL}}{\text{IRRATIONAL}} + \frac{\text{IRRATIONAL}}{\text{IRRATIONAL}}
\]

Case 2: \( b^2 - 4ac \) is negative:

\[
x = \frac{\text{INTEGER}}{\text{INTEGER}} \pm \frac{\text{COMPLEX}}{\text{INTEGER}}
\]

\[
x = \frac{\text{RATIONAL}}{\text{COMPLEX}} + \frac{\text{COMPLEX}}{\text{COMPLEX}}
\]
Example 13. Construct the lowest-degree polynomial given the zeros below:

\[ \sqrt{2}; \frac{1}{3} \]

Since \( \sqrt{2} \) is a zero, \(-\sqrt{2} \) is also a zero.

**Factors:**
1. \((x-\sqrt{2})\)
2. \((x+\sqrt{2})\)
3. \((x-\frac{1}{3})\)  \(\Rightarrow\) GET RID OF FRACTIONS: \((x-\frac{1}{3})^3 \rightarrow 3x-1=0\)

\[
f(x) = (x-\sqrt{2})(x+\sqrt{2})(3x-1) \\
f(x) = (x^2 + \sqrt{2}x - \sqrt{2}x - 2)(3x-1) \\
f(x) = (x^2-2)(3x-1) \\
f(x) = 3x^3 - 6x^2 - 2x - 2
\]

Example 14. Construct the lowest-degree polynomial given the zeros below:

\(4+3i, -\frac{2}{5}\)

Since \(4+3i\) is a zero, \(4-3i\) is also a zero.

**Factors:**
1. \((x-(4+3i))\) = \((x-4-3i)\)
2. \((x-(4-3i))\) = \((x-4+3i)\)
3. \((x-\frac{7}{5})\)  \(\Rightarrow\) GET RID OF FRACTIONS: \((x+\frac{2}{5})^5 \rightarrow 5x+2=0\)

\[
f(x) = (x-4-3i)(x-4+3i)(5x+2) \\
f(x) = ((x-4)-3i)((x-4)+3i)(5x+2) \\
f(x) = [(x-4)^2 -(3i)^2](5x+2) \\
f(x) = [x^2 - 8x + 16 - 9i^2](5x+2) \\
f(x) = (x^2 - 8x + 25)(5x+2) \\
f(x) = 5x^3 + 2x^2 - 40x^2 - 16x + 125x + 50 \\
f(x) = 5x^3 - 38x^2 + 109x + 50
\]