8 Logarithmic and Exponential Functions

8.1 Domain and Range

Exponential Functions

Exponential functions have the form $f(x) = b^x$ for any real number $x$ and constant $b > 0, b \neq 1$.

Graph of Exponential Function: The graph of the parent function, $f(x) = b^x$ is shown below.

We call the two cases exponential $b > 1$ and exponential $0 < b < 1$. 
Characteristics of the Graph of $b^x$

An exponential function of the form $f(x) = b^x$, $b > 0$, $b \neq 1$ has the following characteristics:

- **Horizontal Asymptote at** ________________

- **Domain:** ________________

- **Range:** ________________

- **Vertical Asymptote:** ________________

- **$x$-intercept:** ________________

- **$y$-intercept:** ________________

- **Increasing if** ________________

- **Decreasing if** ________________
Shifts of the Parent Function, \( f(x) = b^x \)

For any constants \( c \) and \( d \), the function \( b^{x+c} + d \) shifts the graph of the parent function \( f(x) = b^x \):

- Vertically _____ units, in the ________________ direction as the sign of _____
- Horizontally _____ units, in the ________________ direction as the sign of _____
- The \( y \)-intercept becomes ________________
- The horizontal asymptote becomes ________________
- The range becomes ________________
- The domain is ________________ (it remains ________________ )
**Example 1.** Determine the domain AND range of the exponential function:

\[ f(x) = -8^{x-1} + 6 \]

**Example 2.** Determine the domain AND range of the exponential function:

\[ f(x) = -8^{x-10} + 4 \]

**Logarithmic Functions**

Logarithmic functions have the form \( \log_b x \) for any real number \( x > 0 \) and constant \( b > 0, \ b \neq 1 \). We read \( \log_b x \) as "the logarithm with base \( b \) of \( x \)."

**Note 1.** We have that \( y = \log_b x \) is equivalent to \( x = b^y \). So, the logarithmic function \( y = \log_b x \) is the \( \boxed{\text{inverse}} \) of the exponential function \( y = b^x \).
Graph of Logarithm Function: The graph of the parent function, \( f(x) = \log_b(x) \) is shown below:

Characteristics of the Graph of \( \log_b(x) \)

For any real number \( x \) and constant \( b > 0, \, b \neq 1 \), we can see the following characteristics in the graph of \( f(x) = \log_b(x) \):

- Vertical Asymptote at ______________________
- Domain: ______________________
- Range: ______________________
- \( x \)-intercept: ______________________
- Key point: ______________________
- \( y \)-intercept: ______________________
- Increasing if ______________________
- Decreasing if ______________________
**Note 2.** For any constant, \( c \), the function \( f(x) = \log_b(x + c) \) shifts the graph of \( \log_b(x) \) by ____ units to the ______________ if \( c > 0 \) and by ____ units to the ______________ if \( c < 0 \). When we shift the graph of \( \log_b(x) \) to the right and left, we must also shift the ______________ of the function.

**How to Determine the Domain of Logarithm Functions**

Recall that the domain of the parent function, \( \log_b(x) \) is ______________. Since the graph of \( \log_b(x + c) \) shifts the graph of \( \log_b(x) \) to the right and left, we must also shift the domain (and vertical asymptote). The function \( \log_b(x + c) \) has a vertical asymptote at ______________, so the domain of \( \log_b(x + c) \) is ______________.

**Note 3.** Another way to consider finding the domain of \( \log_b(x + c) \) is to solve ______________.

**Definition**

A ______________ ______________ is a logarithm with base 10. We write \( \log_{10}(x) \) as ______________. The common logarithm of a positive number \( x \) satisfies the following definition:

For \( x > 0 \), ______________

**Note 4.** Since the graph of a logarithmic function \( \log_b(x + c) + d \) does not have any ______________ asymptotes, the range is ______________ (it remains ______________).
Example 3. Determine the domain AND range of the logarithmic function:

\[ f(x) = \log(x - 5) + 7 \]

Example 4. Determine the domain AND range of the logarithmic function:

\[ f(x) = -\log(x - 9) + 10 \]

Note 5. The domain of logarithmic functions tells us that we cannot take the logarithm of a ________________ number. We also cannot take the logarithm of _____.

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8.2 Convert Between Forms

Relationship Between Logarithmic Functions and its Corresponding Exponential Form

We can express the relationship between logarithmic functions and its corresponding exponential form as follows:

How to Convert From Logarithmic Form to Exponential Form

1. Examine the equation $y = \log_b (x)$ and identify ____, ____, and ____.

2. Rewrite $y = \log_b (x)$ as ________________.

Note 6. To convert from exponential form to logarithmic form, follow the same steps above in reverse.

Example 5. Convert the function below from logarithmic form to exponential form:

$$y = \log_7 (9)$$
Example 6. Convert the function below from logarithmic form to exponential form:

\[ y = \log_{10}(x - 6) + 1 \]

Example 7. Convert the function below from exponential form to logarithmic form:

\[ y = 10^{x-4} + 1 \]

Note 7. Changing between forms is most helpful when trying to solve logarithmic equations.

Example 8. Solve the logarithmic equation below:

\[ \log_{4}(4x) = 9 \]
Example 9. Solve the logarithmic equation below:

\[ \log_3 (4x - 6) + 8 = -\frac{2}{3} \]

Example 10. Solve \( y = \log_4 (64) \) without using a calculator.

Note 8. Recall that \( \pi \approx 3.14 \). Similarly, we can define a new irrational number, \( e \approx 2.718281828... \)

Definition

The function given by \( f(x) = e^x \) is called the ____________

_____________________________ ______________ with natural base \( e \).

Definition

A _______________ _______________ is a logarithm with base \( e \). We write \( \log_e (x) \) as
The natural logarithm of a positive number $x$ satisfies the following definition:

For $x > 0$, $\ln(x) = \ln(e^x)$.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = \ln(x)$ for all $x$, and $e^{\ln(x)} = x$ for all $x > 0$. 
8.3 Properties of Logs

**Basic Logarithm Properties**

Two basic properties of logarithms are as follows:

\[ \log_b (1) = \quad \]  

\[ \log_b (b) = \quad \]

**One-to-One Property**

The one-to-one property for logarithms states that \( \log_b M = \log_b N \) if and only if

**The Product Rule for Logarithms**

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms:

\[ \quad \]

**Example 11.** Expand \( \log_2 xy \)
Example 12. Expand $\log_3 (30x(3x + 4))$

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm of a quotient by rewriting it as a difference of individual logarithms:

Example 13. Expand $\log_2 \left( \frac{x}{y} \right)$
Example 14. Expand $\log_3 \left( \frac{7x^2 + 21x}{7x(x-1)(x-2)} \right)$
The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as a product of the exponent times the logarithm of a base:

Example 15. Expand \( \log_2 x^5 \)
Example 16. Use properties of logarithms to simplify the expression below:

\[
\log \left( \sqrt{\frac{8x^7y^3}{z^4}} \right)
\]
Example 17. Use properties of logarithms to simplify the expression below:

$$\log \left( \sqrt[4]{\frac{4x^4y^5}{z^5}} \right)$$
Example 18. Use properties of logarithm functions to solve the logarithmic equation below:

\[ 6 = \ln \left( \sqrt{\frac{4}{e^x}} \right) \]
8.4 Solve Exponential Functions

One-to-One Property of Exponential Functions

For any algebraic expressions $S$ and $T$, and any positive real number $b \neq 1$, $b^S = b^T$ if and only if 

Using the One-to-One Property to Solve Exponential Equations

1. Rewrite each side of the equation as a power with a 

2. Use the rules of exponents to simplify so that the resulting equation has the form 

3. Use the One-to-One property to set the exponents equal.

4. Solve the resulting equation, $S = T$ for the unknown.

Example 19. Solve the exponential equation below:

$$2^{-5x-6} = 2^{4x+4}$$
Example 20. Solve the exponential equation below:

\[ 4^{5x-3} = 2^{-4x+5} \]

Example 21. Solve the exponential equation below:

\[ \left( \frac{1}{4} \right)^{4x-2} = 2^{-6x+6} \]
**Note 9.** Using the one-to-one property is very useful, but sometimes we will be given an equation in which the one-to-one property cannot be applied.

### Solving Exponential Equations Using Logarithms

1. Take the logarithm of both sides of the equation.

2. If one of the terms in the equation has base 10, use the ________________
   ____________________.

3. If neither of the terms in the equation has base 10, then use the ________________
   ____________________.

4. Use the properties of logarithms to solve for the unknowns.

**Example 22.** Solve the exponential equation below:

\[ 6^{-5x-6} = 5^{6x+4} \]
Example 23. Solve the exponential equation below:

\[27^{-6x-3} = \left(\frac{1}{16}\right)^{2x+3}\]