8 Logarithmic and Exponential Functions

8.1 Domain and Range

**Exponential Functions**

Exponential functions have the form \( f(x) = b^x \) for any real number \( x \) and constant \( b > 0, b \neq 1 \).

**Graph of Exponential Function:** The graph of the parent function, \( f(x) = b^x \) is shown below.

We call the two cases exponential \( f(x) = b^x \) \( b > 1 \) and exponential \( f(x) = b^x \) \( 0 < b < 1 \).
Characteristics of the Graph of $b^x$

An exponential function of the form $f(x) = b^x$, $b > 0$, $b \neq 1$ has the following characteristics:

- Horizontal Asymptote at ________________
- Domain: ________________
- Range: ________________
- Vertical Asymptote: ________________
- $x$-intercept: ________________
- $y$-intercept: ________________
- Increasing if ________________
- Decreasing if ________________
Shifts of the Parent Function, $f(x) = b^x$

For any constants $c$ and $d$, the function $b^{x+c} + d$ shifts the graph of the parent function $f(x) = b^x$:

- Vertically _____ units, in the __________________ direction as the sign of _____
- Horizontally _____ units, in the __________________ direction as the sign of _____
- The $y$-intercept becomes ________________
- The horizontal asymptote becomes ________________
- The range becomes ________________
- The domain is ________________ (it remains ________________ )
Example 1. Determine the domain AND range of the exponential function:

\[ f(x) = -8^{x-4} + 6 \]

Example 2. Determine the domain AND range of the exponential function:

\[ f(x) = -8^{x-10} + 4 \]

Logarithmic Functions

Logarithmic functions have the form \( \log_b x \) for any real number \( x > 0 \) and constant \( b > 0, \, b \neq 1 \). We read \( \log_b x \) as "the logarithm with base \( b \) of \( x \)."

Note 1. We have that \( y = \log_b x \) is equivalent to \( x = b^y \). So, the logarithmic function \( y = \log_b x \) is the \( \text{____________} \) of the exponential function \( y = b^x \).
Graph of Logarithm Function: The graph of the parent function, \( f(x) = \log_b(x) \) is shown below:

![Graph of Logarithm Function](image)

Characteristics of the Graph of \( \log_b(x) \)

For any real number \( x \) and constant \( b > 0, \ b \neq 1 \), we can see the following characteristics in the graph of \( f(x) = \log_b(x) \):

- Vertical Asymptote at _________________
- Domain: _________________
- Range: _________________
- \( x \)-intercept: _________________
- Key point: _________________
- \( y \)-intercept: _________________
- Increasing if _________________
- Decreasing if _________________
Note 2. For any constant, $c$, the function $f(x) = \log_b (x + c)$ shifts the graph of $\log_b (x)$ by ____ units to the ________________ if $c > 0$ and by ____ units to the ________________ if $c < 0$. When we shift the graph of $\log_b (x)$ to the right and left, we must also shift the ________________ of the function.

How to Determine the Domain of Logarithm Functions

Recall that the domain of the parent function, $\log_b (x)$ is _________________. Since the graph of $\log_b (x + c)$ shifts the graph of $\log_b (x)$ to the right and left, we must also shift the domain (and vertical asymptote). The function $\log_b (x + c)$ has a vertical asymptote at ________________, so the domain of $\log_b (x + c)$ is ________________.

Note 3. Another way to consider finding the domain of $\log_b (x + c)$ is to solve _________________.

Definition

A ________________ ________________ is a logarithm with base 10. We write $\log_{10} (x)$ as ________________. The common logarithm of a positive number $x$ satisfies the following definition:

For $x > 0$, ________________

Note 4. Since the graph of a logarithmic function $\log_b (x + c) + d$ does not have any ________________ asymptotes, the range is ________________ (it remains ________________).
Example 3. Determine the domain AND range of the logarithmic function:

\[ f(x) = \log(x - 5) + 7 \]

Example 4. Determine the domain AND range of the logarithmic function:

\[ f(x) = -\log(x - 9) + 10 \]

Note 5. The domain of logarithmic functions tells us that we cannot take the logarithm of a 
____________________ number. We also cannot take the logarithm of _____.

8.2 Convert Between Forms

Relationship Between Logarithmic Functions and its Corresponding Exponential Form

We can express the relationship between logarithmic functions and its corresponding exponential form as follows:

How to Convert From Logarithmic Form to Exponential Form

1. Examine the equation $y = \log_b(x)$ and identify $\text{___}, \text{___}$, and $\text{___}$.

2. Rewrite $y = \log_b(x)$ as $\text{____________}$.

Note 6. To convert from exponential form to logarithmic form, follow the same steps above in reverse.

Example 5. Convert the function below from logarithmic form to exponential form:

$$y = \log_7(9)$$
Example 6. Convert the function below from logarithmic form to exponential form:

\[ y = \log_{10} (x - 6) + 1 \]

Example 7. Convert the function below from exponential form to logarithmic form:

\[ y = 10^{x-4} + 1 \]

Note 7. Changing between forms is most helpful when trying to solve logarithmic equations.

Example 8. Solve the logarithmic equation below:

\[ \log_4 (4x) = 9 \]
Example 9. Solve the logarithmic equation below:

\[ \log_3 (4x - 6) + 8 = -\frac{2}{3} \]

Example 10. Solve \( y = \log_4 (64) \) without using a calculator.

Note 8. Recall that \( \pi \approx 3.14 \). Similarly, we can define a new irrational number, \( e \approx 2.718281828 \ldots \)

Definition

The function given by \( f(x) = e^x \) is called the ________________
_________________________ ________________ with natural base \( e \).

Definition

A ________________ ________________ is a logarithm with base \( e \). We write \( \log_e (x) \) as
The natural logarithm of a positive number $x$ satisfies the following definition:

For $x > 0$, ________________________________.

Since the functions $y = e^x$ and $y = \ln(x)$ are inverse functions, $\ln(e^x) = \underline{\hspace{2cm}}$ for all $x$, and $e^{\ln(x)} = \underline{\hspace{2cm}}$ for all $x > 0$. 
8.3 Properties of Logs

Basic Logarithm Properties

Two basic properties of logarithms are as follows:

\[ \log_b(1) = \text{__________} \]

\[ \log_b(b) = \text{__________} \]

One-to-One Property

The one-to-one property for logarithms states that \( \log_b M = \log_b N \) if and only if \( \text{__________} \)

The Product Rule for Logarithms

The product rule for logarithms can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms:

\[ \text{__________________________} \]

Example 11. Expand \( \log_2 xy \)
Example 12. Expand $\log_3 (30x(3x + 4))$

The Quotient Rule for Logarithms

The quotient rule for logarithms can be used to simplify a logarithm of a quotient by rewriting it as a difference of individual logarithms:

Example 13. Expand $\log_2 \left( \frac{x}{y} \right)$
Example 14. Expand \( \log_3 \left( \frac{7x^2 + 21x}{7x(x - 1)(x - 2)} \right) \)
The Power Rule for Logarithms

The power rule for logarithms can be used to simplify the logarithm of a power by rewriting it as a product of the exponent times the logarithm of a base:

Example 15. Expand \( \log_2 x^5 \)
Example 16. Use properties of logarithms to simplify the expression below:

$$\log \left( \frac{\sqrt{8x^7y^3}}{z^4} \right)$$
Example 17. Use properties of logarithms to simplify the expression below:

$$\log \left( \frac{\sqrt[4]{x^4y^5}}{z^5} \right)$$
Example 18. Use properties of logarithm functions to solve the logarithmic equation below:

$$6 = \ln \left( \sqrt{\frac{4}{e^x}} \right)$$
8.4 Solve Exponential Functions

One-to-One Property of Exponential Functions

For any algebraic expressions $S$ and $T$, and any positive real number $b \neq 1$, $b^S = b^T$ if and only if ___________.

Using the One-to-One Property to Solve Exponential Equations

1. Rewrite each side of the equation as a power with a ________________ ________________.

2. Use the rules of exponents to simplify so that the resulting equation has the form

   __________ = __________.

3. Use the One-to-One property to set the exponents equal.

4. Solve the resulting equation, $S = T$ for the unknown.

Example 19. Solve the exponential equation below:

$$2^{-5x-6} = 2^{4x+4}$$
Example 20. Solve the exponential equation below:

\[ 4^{5x-3} = 2^{-4x+5} \]

Example 21. Solve the exponential equation below:

\[ \left(\frac{1}{4}\right)^{4x-2} = 2^{-6x+6} \]
Note 9. Using the one-to-one property is very useful, but sometimes we will be given an equation in which the one-to-one property cannot be applied.

**Solving Exponential Equations Using Logarithms**

1. Take the logarithm of both sides of the equation.

2. If one of the terms in the equation has base 10, use the ________________
   ________________.

3. If neither of the terms in the equation has base 10, then use the ________________
   ________________.

4. Use the properties of logarithms to solve for the unknowns.

**Example 22.** Solve the exponential equation below:

\[
6^{-5x-6} = 5^{6x+4}
\]
Example 23. Solve the exponential equation below:

\[ 2^{7-6x-3} = \left( \frac{1}{16} \right)^{2x+3} \]