Module 9L Lecture Notes

MAC1105
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9 Operations on Functions

9.1 Domain

Operations on Functions

For two functions \(f(x)\) and \(g(x)\) with real number outputs, we define new functions \(f + g, f - g, fg,\) and \(\frac{f}{g}\) by:

\[
\begin{align*}
(f + g)(x) &= f(x) + g(x) \\
(f - g)(x) &= f(x) - g(x) \\
(fg)(x) &= f(x)g(x) \\
\left(\frac{f}{g}\right)(x) &= \frac{f(x)}{g(x)} \text{ where } g(x) \neq 0
\end{align*}
\]

Example 1. Find and simplify the functions \((g - f)(x)\) and \(\left(\frac{g}{f}\right)(x)\), given \(f(x) = x - 1\) and \(g(x) = x^2 - 1\).

1. \((g - f)(x) = g(x) - f(x) = (x^2 - 1) - (x - 1) = x^2 - 1 - x + 1 = x^2 - x\)

2. \(\left(\frac{g}{f}\right)(x) = \left(\frac{g(x)}{f(x)}\right) = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1} = x + 1, \quad x \neq 1\)
Domain of Algebra of Functions

Let \( f \) and \( g \) be two functions with domains \( A \) and \( B \). Then,

<table>
<thead>
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<th>Name</th>
<th>Definition</th>
<th>Domain</th>
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<tr>
<td>( f \pm g )</td>
<td>( f(x) \pm g(x) )</td>
<td>( A \cap B )</td>
</tr>
<tr>
<td>( fg )</td>
<td>( f(x)g(x) )</td>
<td>( A \cap B )</td>
</tr>
<tr>
<td>( \frac{f}{g} )</td>
<td>( \frac{f(x)}{g(x)} )</td>
<td>( A \cap B \cap {x : g(x) \neq 0} )</td>
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Example 2. Determine the domain of each function below and then determine the domain of $f + g$, $fg$, and $\frac{f}{g}$:

$$f(x) = -5x^2 - 6x + 3$$
$$g(x) = \sqrt{3x - 4}$$

**Domain of $f(x)$:** $(-\infty, \infty) \quad \Rightarrow \quad \text{Domain of } f(x) \text{ is } (-\infty, \infty)$

**Domain of $g(x)$:** $3x - 4 > 0$

$$3x > 4 \quad \Rightarrow \quad x > \frac{4}{3} \quad \Rightarrow \quad \text{Domain of } g(x) \text{ is } \left[\frac{4}{3}, \infty\right) \quad \Rightarrow \quad \text{Domain of } g(x) \text{ is } \left[\frac{4}{3}, \infty\right)$$

1. **Domain of $f(x) + g(x)$ and $f(x)g(x)$ is $A \cap B$:**

$$(-\infty, \infty) \cap \left[\frac{4}{3}, \infty\right) = \left[\frac{4}{3}, \infty\right)$$

2. **Domain of $\frac{f(x)}{g(x)}$ is $A \cap B \cap \{x : g(x) \neq 0\}$:**

$g(x) = 0$ when $x = \frac{4}{3}$, so $A \cap B \cap \{x : g(x) \neq 0\} = \left(\frac{4}{3}, \infty\right)$

We cannot have $x = \frac{4}{3}$
Example 3. Determine the domain of each function below and then determine the domain of \(f + g\), \(fg\), and \(\frac{f}{g}\):

\[
f(x) = -5x^3 + 5x^2 - 5x - 6
\]

\[
g(x) = \sqrt{4x - 6}
\]

**DOMAIN OF** \(f(x)\): \((-\infty, \infty)\)

\[\text{DOMAIN OF } g(x): 4x - 6 > 0 \implies 4x > 6 \implies x > \frac{3}{2} \implies \text{DOMAIN OF } g(x) \text{ IS } \left[\frac{3}{2}, \infty \right) \]

1. **DOMAIN OF** \(f(x) + g(x) \text{ AND } f(x)g(x) \text{ IS } A \cap B:\)

\[
(-\infty, \infty) \cap \left[\frac{3}{2}, \infty \right) = \left[\frac{3}{2}, \infty \right)
\]

2. **DOMAIN OF** \(\frac{f(x)}{g(x)} \text{ IS } A \cap B \cap \{x: g(x) \neq 0\}:

\[
\& \ g(x) = 0 \text{ WHEN } x = \frac{3}{2}, \text{ SO } A \cap B \cap \{x: g(x) \neq 0\} = \left(\frac{3}{2}, \infty \right)
\]

\(\& \) WE CANNOT HAVE \(x = \frac{3}{2}\)
Example 4. Determine the domain of each function below and then determine the domain of \( f + g, fg, \) and \( \frac{f}{g} \):

\[ f(x) = 6x^3 + 6x^2 - 3x + 6 \]
\[ g(x) = -\frac{1}{4x + 3} \]

**DOMAIN OF \( f(x) \):** \((-\infty, \infty)\)

**DOMAIN OF \( g(x) \):** ALL REAL NUMBERS EXCEPT \( 4x + 3 = 0 \):

\[ 4x + 3 = 0 \]
\[ 4x = -3 \]
\[ x = -\frac{3}{4} \]

\( \Rightarrow \) **DOMAIN OF \( g(x) \) IS ALL REAL NUMBERS EXCEPT \( x = -\frac{3}{4} \) \((-\infty, -\frac{3}{4}) \cup (-\frac{3}{4}, \infty)\) **B**

1. **DOMAIN OF \( f(x) + g(x) \) AND \( f(x)g(x) \) IS \( A \cap B \):

\[ (-\infty, \infty) \cap \left[ (-\infty, -\frac{3}{4}) \cup (-\frac{3}{4}, \infty) \right] = (-\infty, -\frac{3}{4}) \cup (-\frac{3}{4}, \infty) \]

2. **DOMAIN OF \( \frac{f(x)}{g(x)} \) IS \( A \cap B \cap \{ x : g(x) \neq 0 \} \):

\( \ast \) \( g(x) \) DOES NOT EVER EQUAL 0 HERE! SO, \( A \cap B \cap \{ x : g(x) \neq 0 \} = A \cap B \)

\( \Rightarrow \) **DOMAIN IS SAME AS 1:**

\[ (-\infty, -\frac{3}{4}) \cup (-\frac{3}{4}, \infty) \]
9.2 Composition

**Definition**

The process of combining functions so that the output of one function becomes the input of another is known as **composition of functions**. For any input $x$ and functions $f$ and $g$, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x))$$

The domain of the composite function $f \circ g$ is all $x$ such that $x$ is in the domain of $g$ and $g(x)$ is in the domain of $f$.

**Note 1.** It is important to keep in mind the order of operations when composing functions. That is, $(f \circ g)(x) = f(g(x))$ means that the function $f$ takes $g(x)$ as an input and yields an output of $f(g(x))$. 


Example 5. For the two functions below, evaluate \((f \circ g)(-3)\):

\[f(x) = 4x^2 - 4x + 4\]

\[g(x) = -3x^2 - 3x + 4\]

\((f \circ g)(-3) = f(g(-3))\)

1. Find \(g(-3)\):

\[g(x) = -3x^2 - 3x + 4\]

\[g(-3) = -3(-3)^2 - 3(-3) + 4\]

\[= -3(9) + 9 + 4\]

\[= -27 + 9 + 4\]

\[= -14\]

\[\Rightarrow g(-3) = -14\]

2. Find \(f(g(-3))\):

\[f(g(-3)) = f(-14)\]

\[f(x) = 4x^2 - 4x + 4\]

\[f(-14) = 4(-14)^2 - 4(-14) + 4\]

\[= 4(196) + 56 + 4\]

\[= 784 + 56 + 4\]

\[= 844\]

\[\Rightarrow (f \circ g)(-3) = f(g(-3)) = f(-14) = 844\]
Example 6. For the two functions below, evaluate \((f \circ g)(3)\) and \((g \circ f)(3)\):

\[
f(x) = 5x^2 - 5x - 3
\]

\[
g(x) = \sqrt{5x - 4}
\]

1. \((f \circ g)(3) = f(g(3))\)
\[
\star g(3) = \sqrt{5(3) - 4} = \sqrt{15 - 4} = \sqrt{11}
\]
\[
\Rightarrow (f \circ g)(3) = f(g(3)) = f(\sqrt{11})
\]
\[
\star f(\sqrt{11}) = 5(\sqrt{11})^2 - 5(\sqrt{11}) - 3
\]
\[
= 5(11) - 5\sqrt{11} - 3
\]
\[
= 55 - 5\sqrt{11} - 3
\]
\[
= 52 - 5\sqrt{11}
\]
\[
\Rightarrow (f \circ g)(3) = 52 - 5\sqrt{11}
\]

2. \((g \circ f)(3) = g(f(3))\)
\[
\star f(3) = 5(3)^2 - 5(3) - 3
\]
\[
= 5(9) - 15 - 3
\]
\[
= 45 - 15 - 3
\]
\[
= 27
\]
\[
\Rightarrow (g \circ f)(3) = g(f(3)) = g(27)
\]
\[
\star g(27) = \sqrt{5(27) - 4}
\]
\[
= \sqrt{135 - 4}
\]
\[
= \sqrt{131}
\]
\[
\Rightarrow (g \circ f)(3) = \sqrt{131}
\]

Note 2. The example above shows that function compositions is not \textbf{Commutative}. That is, \((f \circ g)(x) \neq (g \circ f)(x)\).

Note 3. The product of functions \(fg\) is not the same as the function composition \(f(g(x))\) because \(f(g(x)) \neq f(x)g(x)\).
9.3 One-to-One

**Definition**
A function \( f \) is a **One-to-One** function if each value of the dependent variable \((y)\) corresponds to exactly one value of the independent variable \((x)\).

**Note 4.** If a function \( f \) is a set of ordered pairs, then \( f \) is one-to-one of no two ordered pairs have the same second element. That is, if each \( y \) has only one \( x \).

**Horizontal Line Test**
A function \( f \) is one-to-one if and only if any horizontal line intersects the graph of \( f \) at most once.

**Example 7.** Is the following graph one-to-one?
Example 8. Is the following graph one-to-one?

Algebraically Determine if a Function is One-to-One

To show that a function is one-to-one, you can show that $f(y) = f(x)$ if and only if $y = x$. 

YES, THIS FUNCTION IS 1-1 BECAUSE IT PASSES THE HORIZONTAL LINE TEST!
Example 9. Is the following function one-to-one?

\[ f(x) = 3x + 4 \]

\[ \text{GRAPH:} \]

\[ \Rightarrow \text{YES, 1-1} \]

\[ \text{(IT PASSES HORIZONTAL LINE TEST)} \]

Example 10. Is the following function one-to-one?

\[ f(x) = x^3 - 1 \]

\[ \text{GRAPH SKETCH:} \]

\[ \text{YES, 1-1} \]

\[ \text{PASSES HORIZONTAL LINE TEST!} \]
Example 11. Is the following function one-to-one?

\[ f(x) = \sqrt{x - 1} + 3 \]

\( f(h, k) = (1, 3) \)

**Graph:**

\( f(x) = (x - 1)^2 + 3 \) is not 1-1 because it does not pass the horizontal line test.
9.4 Inverse

Definition

For any one-to-one function \( f(x) = y \), a function \( f^{-1}(x) \) is an **Inverse Function** of \( f \) if \( f^{-1}(y) = x \). This can also be written as \( f^{-1}(f(x)) = x \) for all \( x \) in the domain of \( f \). It also follows that \( f(f^{-1}(x)) = x \) for all \( x \) in the domain of \( f^{-1} \).

**Note 5.** Not every function has an inverse, and \( f^{-1}(x) \neq \frac{1}{f(x)} \). Given a one-to-one function, \( f \), the inverse of the coordinate pair \( (x, f(x)) \) is \( (f(x), x) \).

**Example 12.** For a particular one-to-one function \( f(2) = 4 \) and \( f(5) = 12 \), what are the corresponding input and output values for the inverse function?

\[
\begin{align*}
  f(2) &= 4 \Rightarrow f^{-1}(4) = 2 & \quad (2, 4) & \quad \rightarrow (4, 2) \\
  f(5) &= 12 \Rightarrow f^{-1}(12) = 5 & \quad (5, 12) & \quad \rightarrow (12, 5)
\end{align*}
\]

**How to Determine if Two Functions \( f(x) \) and \( g(x) \) are Inverses of Each Other**

1. Determine whether \( f(g(x)) = x \) or \( g(f(x)) = x \).

2. If either statement is true, then both are true, and \( g = f^{-1} \) and \( f = g^{-1} \). If either statement is false, then both are false, and \( g \neq f^{-1} \) and \( f \neq g^{-1} \).

**Domain and Range of Inverse Functions**

The **Range** of a function \( f(x) \) is the domain of the inverse function \( f^{-1}(x) \). The **Domain** of \( f(x) \) is the range of \( f^{-1}(x) \).
How to Find the Domain and Range of an Inverse Function

1. If the original function is one-to-one, write the range of the original function as the **DOMAIN** of the inverse function.

2. If the original function is one-to-one, write the domain of the original function as the **RANGE** of the inverse function.

3. If the domain of the original function needs to be restricted to make it one-to-one, then this restricted domain becomes the **RANGE** of the inverse function.

How to Determine the Inverse of a Function

1. Check that \( f \) is a **ONE** - TO - **ONE** function.

2. Solve for \( x \).

3. Interchange \( x \) and \( y \).
**Example 13.** Determine if the following function is one-to-one. If the function is one-to-one, find the inverse and define the domain on which the inverse is valid:

\[ f(x) = (4x - 5)^3 - 4 \]

1. Is \( f \) 1-1?
   
   **YES!** In general, \( x^3 \) is 1-1

2. \( y = (4x - 5)^3 - 4 \)  
   
   \( x = (4y - 5)^3 - 4 \)
   
   \( x - 4 = (4y - 5)^3 \)
   
   \( (x - 4)^{\frac{1}{3}} = [(4y - 5)^3]^{\frac{1}{3}} \)

   \( \sqrt[3]{x-4} = 4y - 5 \)

   \( \sqrt[3]{x-4} + 5 = 4y \)

   \( \frac{\sqrt[3]{x-4} + 5}{4} = y \)

   \( \Rightarrow f^{-1}(x) = \frac{1}{4} \sqrt[3]{x-4} + \frac{5}{4} \)

**RANGE OF** \( f(x) = (4x - 5)^3 - 4 \) **IS** \((-\infty, \infty)\)

**\Rightarrow DOMAIN OF** \( f^{-1}(x) \) **IS** \((-\infty, \infty)\)
**Example 14.** Determine if the following function is one-to-one. If the function is one-to-one, find the inverse and define the domain on which the inverse is valid:

\[ f(x) = \sqrt{-3x - 5} + 7 \]

1. **Is** \( f(x) \) **1-1?**

   YES! **SQUARE ROOT FUNCTION** \( \Rightarrow \) **1-1**

2. \( y = \sqrt{-3x - 5} + 7 \)

\[ x = \sqrt{-3y - 5} + 7 \]

\[ x - 7 = \sqrt{-3y - 5} \]

\[ (x - 7)^2 = (-3y - 5) \]

\[ x^2 - 14x + 49 = -3y - 5 \]

\[ x^2 - 14x + 54 = -3y \]

\[ \frac{x^2 - 14x + 54}{-3} = y \]

\[ \Rightarrow f^{-1}(x) = -\frac{1}{3}x^2 + \frac{14}{3}x - 18 \]

**Range is** \([7, \infty)\) **or** \(y \geq 7\)

**Domain of** \( f^{-1}(x) \) **is the range of** \( f(x) \):

**Range of** \( f(x) \) **is** \([7, \infty)\)

\[ \Rightarrow \text{Domain of } f^{-1}(x) \text{ is } [7, \infty) \]
Example 15. Determine if the following function is one-to-one. If the function is one-to-one, find the inverse and define the domain on which the inverse is valid:

\[ f(x) = (2x + 7)^2 + 2 \]

1. Is \( f \) one-to-one?

   **NO! PARABOLAS ARE NOT 1-1**

   \[ \implies \text{NO INVERSE FUNCTION EXISTS!} \]