Lecture 25: Section 4.1
Angles and Their Measure

Angle - initial side, terminal side, vertex

Standard position of an angle

Positive and negative angles

Coterminal angle

Central angle

Radians

Complementary and supplementary angles

Degree measure and radian measure

Arc length, \( s \)

Area of a sector

Linear speed

Angular speed
An **angle** is formed by rotating a ray around its endpoint. The starting position of the ray is the **initial side** of the angle, and the position after rotation is the **terminal side** of the angle. The endpoint of the ray is called the **vertex**.

An angle $\theta$ is said to be in **standard position** if its vertex is in the origin and its initial side coincides with the positive $x$-axis.
If the rotation is in the counterclockwise direction, the angle is **positive**; if the rotation is clockwise, the angle is **negative**.

Angles $\alpha$ and $\beta$ are **coterminal angles** if they have the same initial and terminal sides.

A **central angle** is an angle whose vertex is at the center of a circle.
Radian Measure

**Def.** One radian is a measure of the central angle that intercepts an arc whose length is equal to the radius. Algebraically, this means that

\[ \theta = \frac{s}{r} \]

where \( \theta \) is measured in radians.

**NOTE:** For the angle \( \theta = 1 \) revolution:

The length of the arc (circumference) \( s = 2\pi r \)

Therefore, \( \theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \).

We have, 

1 revolution = \( 2\pi \) radians
NOTE: If $0 \leq \theta \leq 2\pi$, the standard position of the angle $\theta$ in the Cartesian coordinate system is shown below:

\[ 3\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} \]

NOTE: Given an angle $\theta$, the coterminal angles to $\theta$ are

\[ \theta + 2n\pi \]

where $n$ is an integer.

ex. Find the angle with the smallest positive measure that is coterminal with $\theta = -\frac{21\pi}{4}$.

\[
\begin{align*}
\text{n=1:} & \quad -\frac{21\pi}{4} + 2\pi = -\frac{21\pi}{4} + \frac{8\pi}{4} = -\frac{13\pi}{4} \\
\text{n=2:} & \quad -\frac{21\pi}{4} + 2\pi(2) = -\frac{21\pi}{4} + \frac{16\pi}{4} = -\frac{5\pi}{4} \\
\text{n=3:} & \quad -\frac{21\pi}{4} + 2\pi(3) = -\frac{21\pi}{4} + \frac{24\pi}{4} = \frac{3\pi}{4}
\end{align*}
\]

Checkpoint: Lecture 25, problem 1
**Def.** Given two positive angles $\alpha$ and $\beta$,

1. $\alpha$ and $\beta$ are **complementary** if $\frac{\pi}{2}$ AND 90° ARE THE SAME THING! $\frac{\pi}{2} + \beta = \frac{\pi}{2}$ or $\frac{\pi}{2} + \beta = 90°$

2. $\alpha$ and $\beta$ are **supplementary** if $\frac{\pi}{2} + \beta = \pi$ or $\frac{\pi}{2} + \beta = 180°$

**ex.** Find the complement and supplement of the angle $\theta = \frac{\pi}{7}$. LET $\beta = \text{ANGLE WE ARE LOOKING FOR}$

**Complement:**

$$\frac{\pi}{2} + \beta = \frac{\pi}{2} \Rightarrow \beta = \frac{\pi}{2} - \frac{\pi}{2} = \frac{2\pi}{14} - \frac{2\pi}{14} = \frac{5\pi}{14}$$

**Supplement:**

$$\frac{\pi}{2} + \beta = \pi \Rightarrow \beta = \pi - \frac{\pi}{2} = \frac{2\pi}{7} - \frac{\pi}{2} = \frac{6\pi}{7}$$

Checkpoint: Lecture 25, problem 2
Degree Measure

Another way to measure angles is in terms of **degrees**, denoted by °.

1 counterclockwise revolution = 360°

**NOTE:** 1 revolution = 360° = 2π rad

\[ \frac{180^\circ}{\pi} = 1 \text{ rad} \]

Therefore, we have

\[ 1^\circ = \frac{\pi}{180} \text{ rad} \quad \text{and} \quad 1 \text{ rad} = \frac{180^\circ}{\pi} \]

\[ \frac{90}{3} = 30 \]

\[ \left( \frac{\pi}{2} \right) \left( \frac{1}{3} \right) = \frac{\pi}{6} \]

\[ \left( \frac{\pi}{2} \right) \left( \frac{1}{2} \right) = \frac{\pi}{4} \]
Conversions between Radians and Degrees

Conversions between Radians and Degrees

\[ \text{Radian} \quad \times \quad \frac{180^\circ}{\pi} \quad \rightarrow \quad \text{Degree} \]

\[ \times \quad \frac{\pi}{180^\circ} \quad \rightarrow \quad \text{Radian} \]

**ex.** Convert each angle in degrees to radians.

1) \(60^\circ \cdot \frac{\pi}{180^\circ} = \frac{60\pi}{180} = \frac{\pi}{3}\)

2) \(150^\circ \cdot \frac{\pi}{180^\circ} = \frac{150\pi}{180} = \frac{5\pi}{6}\)

**ex.** Convert each angle in radians to degrees.

1) \(\frac{\pi}{6} \cdot \frac{180^\circ}{\pi} = \frac{\pi 180^\circ}{6\pi} = \frac{180^\circ}{6} = 30^\circ\)

2) \(-\frac{3\pi}{4} \cdot \frac{180^\circ}{\pi} = -\frac{3\pi (180^\circ)}{4\pi} = -135^\circ\)

Checkpoint: Lecture 25, problem 3
Arc Length = \text{PARTIAL CIRCUMFERENCE AROUND CIRCLE}

For a circle of radius $r$, a central angle $\theta$ intercepts an arc of length $s$ given by

\[ s = r\theta \]

where $\theta$ is measured in radians.

\textbf{ex.} A circle has a radius of 6 inches. Find the length of the arc intercepted by a central angle of $120^\circ$.

\[ \theta = 120^\circ \left(\frac{\pi}{180^\circ}\right) \]

\[ \theta = \frac{120\pi}{180} = \frac{2\pi}{3} \]

\[ s = 6 \left(\frac{2\pi}{3}\right) = \frac{12\pi}{3} = 4\pi \text{ INCHES} \]

Checkpoint: Lecture 25, problem 4
Area of a Sector

For a circle of radius $r$, the area $A$ of a sector with central angle $\theta$ is given by

$$A = \frac{1}{2} r^2 \theta$$

where $\theta$ is measured in radians.

ex. A sprinkler sprays water over a distance of 30 feet while rotating through an angle of $150^\circ$. What area of lawn receives water?

![Diagram of sprinkler area]

WANT $\theta$ IN RADIANS!

$$\theta = 150^\circ \left( \frac{\pi}{180^\circ} \right) = \frac{150\pi}{180} = \frac{5\pi}{6}$$

$$A = \frac{1}{2} (30^2) \left( \frac{5\pi}{6} \right)$$

$$A = \frac{900 \cdot 5\pi}{2 \cdot 6} = \frac{375\pi}{1} = 375\pi \text{ sq. ft.}$$

Checkpoint: Lecture 25, problem 5