Lecture 21 (Part I): Cylindrical polar coordinates

Summary of Lecture

1. For a point $P$ in the $\mathbb{R}^3$, $r$ is defined to be the perpendicular distance between $P$ and the $z$ axis, $z$ is the height of $P$ from the $xy$-plane. Drop a perpendicular line from $P$ to $xy$ plane and name the intersection point $Q$. $\theta$ is the angle subtended by the positive $x$ axis and the line $OQ$. Representation of a point $P$ in $\mathbb{R}^3$ in terms of $(r, \theta, z)$ is called the cylindrical polar coordinate representation of $P$.

2. The transformations to cylindrical coordinates is given by
   $$x = r \cos(\theta), \quad y = r \sin(\theta), \quad z = z, \quad dz \, dy \, dx = r \, dz \, dr \, d\theta$$

3. If a point $(x, y, z)$ is in the Cartesian coordinates, its representation in the cylindrical coordinates is given by
   $$r = \sqrt{x^2 + y^2}, \quad z = z, \quad \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases}$$

   Remark: If $x = 0$, then $\theta = \pi/2$ if $y > 0$ and $\theta = 3\pi/2$ if $y = 0$. Also $\theta$ is undefined at the origin.

4. Triple integration over a general region $W = \{(r, \theta, z)|\alpha \leq \theta \leq \beta, g(\theta) \leq r \leq h(\theta), G(r \cos(\theta), r \sin(\theta)) \leq z \leq H(r \cos(\theta), r \sin(\theta))\}$ can be calculated by
   $$\int_{\theta=\alpha}^{\beta} \int_{r=g(\theta)}^{h(\theta)} \int_{z=G(r \cos(\theta), r \sin(\theta))}^{H(r \cos(\theta), r \sin(\theta))} f(r, \theta, z) \, dz \, dr \, d\theta$$

   Observe that the volume element $dV = r \, dz \, dr \, d\theta$.

5. The equation $z = c$ (const.) represents a plane parallel to the $xy$ plane through $(0, 0, c)$, $\theta = \theta_0$(const). represents half planes passing through the origin parallel to the $z$ axis making angle $\theta_0$ with the positive $x$ axis, and the equation $r = R$ (const.) gives cylinders having $z$ axis on the center, and having radius $R$.

6. The volume of the region $W$ given in cylindrical coordinates is
   $$\iiint_W r \, dz \, dr \, d\theta.$$
Exercises

1. Convert the following points from cylindrical to Cartesian coordinates
   (a) \((4, \pi, 4)\)
   (b) \((2, \pi/3, -8)\)
   (c) \((0, \pi/5, 1/2)\)
   (d) \((1, \pi/2, -2)\)

2. Convert the following points from Cartesian to cylindrical coordinates.
   (a) \((1, -1, 1)\)
   (b) \((1, \sqrt{3}, 7)\)
   (c) \((5/\sqrt{2}, 5/\sqrt{2}, 2)\)
   (d) (modified) \((-3, 3\sqrt{3}, 2)\)

3. Find an equation of the form \(z = f(r, \theta)\) in cylindrical coordinates for the following functions
   (a) \(z = x^2 + y^2\)
   (b) \(z = x + y\)
   (c) \(z = \sqrt{x^2 - y^2}\)

4. Convert the following surfaces to cylindrical coordinates
   (a) \(x^2 + y^2 + z^2 = 4\)
   (b) \(x^2 + y^2 = 4\)
   (c) \(x^2 - y^2 = 4\)

5. Use the cylindrical coordinates to find the volume of the region \(W\) between the paraboloids \(z = x^2 + y^2\) and \(z = 8 - x^2 - y^2\).

6. Use cylindrical coordinates to calculate the integral of the function \(f(x, y, z) = z\) over the region above the disk \(x^2 + y^2 = 1\) in the \(xy\)-plane and below the surface \(z = 4 + x^2 + y^2\).

7. Integrate the function \(f(x, y, z) = x^2 + y^2\) over the region bounded by \(x^2 + y^2 \leq 9\), \(0 \leq z \leq 5\) by converting to cylindrical coordinates.

8. Integrate \(f(x, y, z) = z\) over the region \(0 \leq z \leq x^2 + y^2 \leq 9\) using cylindrical polar coordinates.

9. Express the following triple integrals in cylindrical coordinates
   (a) \(\int_{x=-1}^{1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{z=0}^{4} f(x, y, z) \, dz \, dy \, dx\).
   (b) \(\int_{x=0}^{2} \int_{y=0}^{\sqrt{2x-x^2}} \int_{z=0}^{\sqrt{x^2+y^2}} f(x, y, z) \, dz \, dy \, dx\)

10. Use cylindrical coordinates to integrate \(f(x, y, z) = z\) over the intersection of the hemisphere \(x^2 + y^2 + z^2 = 4\), \(z \geq 0\) and the cylinder \(x^2 + y^2 = 1\).
Answers
1. (a) \((-4, 0, 4)\)
   (b) \((1, \sqrt{3}, -8)\)
   (c) \((0, 0, \frac{1}{2})\)
   (d) \((0, 1, -2)\)
2. (a) \((\sqrt{2}, \frac{7\pi}{4}, 1)\)
   (b) \((2, \frac{\pi}{3}, 7)\)
   (c) \((5, \frac{\pi}{4}, 2)\)
   (d) \((6, \frac{5\pi}{6}, 2)\)
3. (a) \(z = r^2\)
   (b) \(z = r \cos(\theta) + r \sin(\theta)\)
   (c) \(z = r \sqrt{\cos(2\theta)}\)
4. (a) \(r^2 + z^2 = 4\)
   (b) \(r = 2\)
   (c) \(r^2 = 4 \sec(2\theta)\)
5. 16\pi
6. \(\frac{61\pi}{6}\)
7. \(\frac{1024\pi}{15}\)
8. \(\frac{729\pi}{6}\)
9. (a) \(\int_{\theta=0}^{2\pi} \int_{r=0}^{1} \int_{z=0}^{4} f(r \cos(\theta), r \sin(\theta), z) r \, dz \, dr \, d\theta\)
   (b) \(\int_{\theta=0}^{\pi/2} \int_{r=0}^{2 \cos(\theta)} \int_{z=0}^{r} f(r \cos(\theta), r \sin(\theta), z) r \, dz \, dr \, d\theta\)
10. 0
1 Lecture 21: (Part 2) Spherical polar coordinates

Summary of Lecture

1. For a point \( P \) in the \( \mathbb{R}^3 \), \( \rho \) is defined to be the distance between \( P \) and the origin \( O \), \( \phi \) is the lesser angle subtended by the \( z \) axis and the line \( OP \). Drop a perpendicular line from \( P \) to \( xy \) plane and name the intersection point \( Q \). \( \theta \) is the angle subtended by the positive \( x \) axis and the line \( OQ \). Representation of a point \( P \) in \( \mathbb{R}^3 \) in terms of \( (\rho, \phi, \theta) \) is called the spherical polar coordinate representation of \( P \).

2. The transformations to spherical coordinates is given by
   \[
   x = \rho \sin(\phi) \cos(\theta) \quad y = \rho \sin(\phi) \sin(\theta) \quad z = \rho \cos(\phi) \quad dz \ dy \ dx = \rho^2 \sin(\phi) \ d\rho \ d\phi \ d\theta
   \]

3. If a point \((x, y, z)\) is in the Cartesian coordinates, its representation in spherical coordinates is given by
   \[
   \rho = \sqrt{x^2 + y^2 + z^2} \quad \phi = \cos^{-1}\left(\frac{z}{\rho}\right) \quad \theta = \begin{cases} \tan^{-1}(y/x) & \text{if } x > 0 \\ \tan^{-1}(y/x) + \pi & \text{if } x < 0 \end{cases}
   \]
   Remark: If \( x = 0 \), then \( \theta = \pi/2 \) if \( y > 0 \) and \( \theta = 3\pi/2 \) if \( y = 0 \). Also \( \theta \) and \( \phi \) are undefined at the origin.

4. For a region \( W \) defined by \( W = \{(\rho, \phi, \theta) | a \leq \theta \leq b, a \leq \phi \leq b, g(\phi, \theta) \leq \rho \leq h(\phi, \theta)\} \), the triple integral \( \iiint_W f \ dV \) is given by
   \[
   \int_{\theta=a}^{\beta} \int_{\phi=a}^{b} \int_{\rho=g(\phi,\theta)}^{h(\phi,\theta)} f(\rho, \phi, \theta) \rho^2 \sin(\phi) \ d\rho \ d\phi \ d\theta
   \]

5. The surface \( \rho = R \) (const) is a sphere of radius \( R \), \( \theta = \theta_0 \) (const) is a vertical half plane and the surface \( \phi = \phi_0 \) (const) is a right circular cone.

6. The relationship between spherical and cylindrical coordinates is given by
   \[
   \theta = \theta, \quad r = \rho \sin(\phi) \quad \phi = \cos^{-1}\left(\frac{z}{\sqrt{\rho^2 + z^2}}\right) \quad z = \rho \cos(\phi)
   \]
   Also the volume of a region \( W \) in spherical coordinates is given by \( \iiint_W \rho^2 \sin(\phi) \ d\rho \ d\phi \ d\theta \).
Exercises

1. Convert from spherical to Cartesian coordinates
   (a) \((3, 0, \pi/2)\)
   (b) \((3, \pi, 0)\)
   (c) \((6, \pi/6, 5\pi/6)\)

2. Convert from Cartesian to spherical coordinates.
   (a) \((\sqrt{3}, 0, 1)\)
   (b) \((1, 1, 1)\)
   (c) \((1/2, -\sqrt{3}/2, 1)\) (modified)

3. Do the following conversions
   (a) \((2, 0, 2)\) from cylindrical to spherical coordinates.
   (b) \((4, 0, \pi/4)\) from spherical to cylindrical coordinates

   For the questions (4) to (8), find the integral of the given function in the given region using spherical coordinates.

4. \(f(x, y, z) = y\) on the region \(x^2 + y^2 + z^2 \leq 1\), \(x, y, z \leq 0\).

5. \(f(x, y, z) = x^2 + y^2\) on the region \(\rho \leq 1\).

6. \(f(x, y, z) = \sqrt{x^2 + y^2 + z^2}\) on the region \(x^2 + y^2 + z^2 \leq 2z\).

7. \(f(x, y, z) = z\) on the region \(0 \leq \theta \leq \pi/3\), \(0 \leq \phi e\pi/2\) and \(1 \leq \rho \leq 2\).

8. \(f(x, y, z) = z(x^2 + y^2 + z^2)^{-3/2}\) over the part of the ball \(x^2 + y^2 + z^2 \leq 16\) defined by \(z \geq 2\).

9. Find the volume of the region lying above the cone \(\phi = \pi/3\) and below the sphere \(\rho = R\).

10. Let \(W\) be the region within the cylinder \(x^2 + y^2 = 2\) between \(z = 0\) and the cone \(z = x^2 + y^2\). Calculate the integral of \(f(x, y, z) = x^2 + y^2\), first using spherical coordinates, and then using cylindrical coordinates.
Answers

1. (a) $(3, 0, 0)$
   (b) $(0, 0, 3)$
   (c) $\left(\frac{3\sqrt{3}}{2}, \frac{3}{2}, -3\sqrt{3}\right)$

2. (a) $(2, 0, \frac{\pi}{3})$
   (b) $\left(\sqrt{3}, \frac{\pi}{4}, \cos^{-1}\left(\frac{1}{3}\right)\right)$
   (c) $(2, \frac{\pi}{3}, \frac{\pi}{6})$

3. (a) $(2\sqrt{2}, \frac{\pi}{4}, 0)$
   (b) $(2\sqrt{2}, 0, 2\sqrt{2})$

4. $-\frac{\pi}{16}$

5. $\frac{8\pi}{15}$

6. $\frac{8\pi}{5}$

7. $\frac{5\pi}{8}$

8. $\pi$

9. $\frac{\pi R^3}{3}$

10. $\frac{8\sqrt{2}\pi}{5}$