Lecture 22: Transformations on the plane, change of variables, the Jacobian determinant

Summary of Lecture

1. Given a transformation $T : x = g(u,v), y = h(u,v)$, where $g$ and $h$ are differentiable functions on the $u$, $v$ plane, the Jacobian determinant is the function defined by

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \left( \frac{\partial x}{\partial u} \right) \left( \frac{\partial y}{\partial v} \right) - \left( \frac{\partial x}{\partial v} \right) \left( \frac{\partial y}{\partial u} \right)$$

2. Let $T : x = g(u,v), y = h(u,v)$ be a transformation that maps a closed, bounded region $R$ in the $uv$-plane to the region $S$ in the $xy$-plane. Assume that $T$ is one to one on the interior of $R$ and $g$ and $h$ have continuous partial derivatives there. For a continuous function $f$, the change of variables formula is given by

$$\iiint_S f(x,y) \, dx \, dy = \iiint_R f(g(u,v),h(u,v)) |J(u,v)| \, dv \, du.$$

3. The Jacobian determinant in the case of 3 variables (so the transformation $x,y,z \mapsto u,v,w$) is given by

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

4. Under similar conditions as in (2) where we now have $x = g(u,v,w), y = h(u,v,w)$ and $z = j(u,v,w)$, the change of variable formula for triple integrals take the form

$$\iiint_S f(x,y,z) \, dz \, dy \, dx = \iiint_R f(g(u,v,w),h(u,v,w),j(u,v,w)) |J(u,v,w)| \, dw \, dv \, du$$

5. Direct calculations show that

(a) The Jacobian for polar coordinate transformation $J(r,\theta) = r$.

(b) The Jacobian for cylindrical coordinate transformation is $J(r,\theta,z) = r$.

(c) The Jacobian for spherical coordinate transformation is $J(\rho,\phi,\theta) = \rho^2 \sin(\phi)$.

Exercises

1. Find the Jacobian for the transformation $x = 3u + 4v, y = u - 2v$

2. Find the Jacobian for the transformation $g(r,t) = r \sin(t), t - \cos(t)$ at $(r,t) = (1,\pi)$

3. Find the Jacobian for the transformation $G(r,\theta) = (r \cos(\theta), r \sin(\theta))$ at the point $(r,\theta) = (4, \pi/6)$.  

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4. Compute \( \iint_D (x + 3y) \, dx \, dy \), where \( D \) is the shaded region in the figure below. Hint: Use the transformation \( G(u, v) = (u - 2v, v) \).

![Diagram](image1)

5. Use the map \( G(u, v) = \left( \frac{u}{v + 1}, \frac{uv}{v + 1} \right) \) to compute \( \iint_D (x + y) \, dx \, dy \), where \( D \) is the shaded region in the figure below.

![Diagram](image2)

6. Show that \( T(u, v) = (u^2 - v^2, 2uv) \) maps the triangle \( D = \{(u, v) : 0 \leq v \leq u \leq 1\} \) to the domain \( S \) bounded by \( x = 0 \), \( y = 0 \) and \( y^2 = 4 - 4x \). Use \( T \) to evaluate \( \iint_D \sqrt{x^2 + y^2} \, dx \, dy \).

7. Calculate \( \iint_D e^{9x^2 + 4y^2} \, dx \, dy \), where \( D \) is the interior of the ellipse \( \frac{x^2}{4} + \frac{y^2}{9} \leq 1 \).

8. Sketch the domain \( D \) bounded by \( y = x^2 \), \( y = \frac{1}{2} x^2 \) and \( y = x \). Use a change of variables with the map \( x = uv \), \( y = u^2 \) to calculate \( \iint_D y^{-1} \, dx \, dy \).

9. Find an appropriate change of variables to evaluate \( \iint_R (x + y)^2 e^{x^2 - y^2} \, dx \, dy \) where \( R \) is the square with vertices \((1, 0), (0, 1), (-1, 0), (0, -1)\).

10. Use the transformation \( x = (r/t) \cos(\theta) \), \( y = (r/t) \sin(\theta) \) and \( z = r^2 \) to find the average value of the function \( f(x, y, z) = \frac{1}{(x^2 + y^2)^{3/2}} \) in the region \( D \) that lies between the paraboloids \( z = x^2 + y^2 \), \( z = 4(x^2 + y^2) \) and the planes \( z = 1 \) and \( z = 4 \).
Answers
1. \(-10\)
2. 1
3. 4
4. 80
5. \(\frac{21}{2}\)
6. \(\frac{56}{45}\)
7. \(\frac{\pi(e^{36} - 1)}{6}\)
8. 1
9. \(\frac{2}{e}\)
10. ☺️