Limits of Multivariable Functions

We are all familiar with the concept of a limit of a function of a single variable. We say that \( \lim_{x \to a} f(x) \) exists if both \( \lim_{x \to a^+} f(x) \) and \( \lim_{x \to a^-} f(x) \) exist and \( \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) \).

The key point is that we can only approach the point \( x = a \) from two directions; from the left or the right.

Now consider a limit for a function of two variables

\[
\lim_{(x,y) \to (a,b)} f(x, y).
\]

In this case the point \( (a, b) \) lies in a plane and there exists an infinite number of ways to approach this point; how can we test them all?
Example: Calculate the limit of the function 
\[ f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \] as \((x, y) \to (0, 0)\) by considering two different approaches: approach along the line \(y = x\) and approach along the line \(y = 0\).

Along the path \(y = x\)

\[
\lim_{x \to 0} f(x, y) = \lim_{x \to 0} \frac{x^2 - x^2}{x^2 + x^2} = \lim_{x \to 0} \frac{0}{2x^2} = 0
\]

Along the path \(y = 0\)

\[
\lim_{x \to 0} f(x, 0) = \lim_{x \to 0} \frac{x^2 - 0^2}{x^2 + 0^2} = 1.
\]

Two different limits on different paths.
The previous example illustrates two important ideas:

First, it is impossible to calculate the limit of a multivariable function by calculating the limit along a given path; the limit may be different along a different path; indeed this approach would only be valid if we could calculate the limit along every path.

Second if a limit for a function is calculated along two different paths and two different values are obtained, the limit does not exist for the function at the given point.

Example: Does a limit exist for the function 
\[ f(x, y) = \frac{(2x + y^2)}{(x^2 - y^2)} \]  

as \( (x, y) \to (0, 0) \).

along the path \( x = 0 \)

\[ \lim_{y \to 0} f(0, y) = \lim_{y \to 0} \frac{2(0) + y^2}{0^2 - y^2} = \lim_{y \to 0} \frac{y^2}{-y^2} = -1. \]

along the path \( x = y^2 \)

\[ \lim_{(x, y) \to (0, 0)} f(x, y) = \lim_{y \to 0} \frac{2y^2 + y^2}{y^4 - y^2} = \lim_{y \to 0} \frac{3y^2}{y^2(y^2 - 1)} \]

\[ = \lim_{y \to 0} \frac{3}{y^2 - 1} = -3. \]

The limit DNE.
So how does one determine if a limit to a function exists at a point, and if it does exist, what is its value?

Consider the following theoretical definition of a limit:

Definition: The function $f(x, y)$ has the limit $L$ as $(x, y) \to (a, b)$, written

$$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

if, given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon$$

whenever

$$0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta.$$
Example: Use the limit definition to prove

\[ \lim_{(x,y) \to (a,b)} x = a. \]

Let \( \varepsilon > 0 \), we must show that we can find a \( \delta > 0 \) such that

\[ |f(x,y) - a| = |x - a| < \varepsilon \text{ for all } (x,y) \in D^*(c, b; \varepsilon) \]

pick \( \delta = \varepsilon \), if \( (x, y) \in D(a, b; \varepsilon) \) then

\[(x - a)^2 + (y - b)^2 < \varepsilon^2 \Rightarrow (x - a)^2 < \varepsilon^2 = |x - a| < \varepsilon.

In other words, for \( \varepsilon > 0 \)

\[ |x - a| < \varepsilon \text{ for all } (x, y) \in D^*(a, b; \varepsilon). \]

\[ \therefore \lim_{(x,y) \to (a,b)} x = a. \]
From the previous example we note that the limit definition does not provide a practical technique for evaluating limits.

Before we can resolve this dilemma of how to calculate limits, we need to develop some other ideas.

We begin with the following theorem:

Theorem 1a

**Limits of Constant and Linear Functions**

Let $a$, $b$, and $c$ be real numbers.

1. Constant function $f(x, y) = c$: $\lim_{(x,y) \to (a,b)} c =$

2. Linear function $f(x, y) = x$: $\lim_{(x,y) \to (a,b)} x =$

3. Linear function $f(x, y) = y$: $\lim_{(x,y) \to (a,b)} y =$

As seen in the previous example which actually verified result two, the $\delta, \epsilon$ definition of the limit is used in proving this theorem.
Theorem 1b

Limit Laws for Functions of Two Variables

Let $L$ and $M$ be real numbers and let
\[ \lim_{(x,y) \to (a,b)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y) \to (a,b)} g(x, y) = M. \]
Assume $c$ is a constant and $m$ and $n$ are integers.

1. **Sum** \[ \lim_{(x,y) \to (a,b)} (f(x, y) + g(x, y)) = \lim_{(x,y) \to (a,b)} f(x, y) + \lim_{(x,y) \to (a,b)} g(x, y) \]

2. **Difference** \[ \lim_{(x,y) \to (a,b)} (f(x, y) - g(x, y)) = \lim_{(x,y) \to (a,b)} f(x, y) - \lim_{(x,y) \to (a,b)} g(x, y) \]

3. **Constant Multiple** \[ \lim_{(x,y) \to (a,b)} cf(x, y) = c \lim_{(x,y) \to (a,b)} f(x, y) \]

4. **Product** \[ \lim_{(x,y) \to (a,b)} f(x, y)g(x, y) = \left( \lim_{(x,y) \to (a,b)} f(x, y) \right) \left( \lim_{(x,y) \to (a,b)} g(x, y) \right) \]

5. **Quotient** \[ \lim_{(x,y) \to (a,b)} \frac{f(x, y)}{g(x, y)} = \frac{\lim_{(x,y) \to (a,b)} f(x, y)}{\lim_{(x,y) \to (a,b)} g(x, y)} \]

provided $M \neq 0$

6. **Power** \[ \lim_{(x,y) \to (a,b)} (f(x, y))^n = \left( \lim_{(x,y) \to (a,b)} f(x, y) \right)^n \]

7. **m/n Power** \[ \lim_{(x,y) \to (a,b)} (f(x, y))^{m/n} = \left( \lim_{(x,y) \to (a,b)} f(x, y) \right)^{m/n} \]

where $m$ and $n$ have no common factors and we assume $L > 0$ if $n$ is even.
We can now return to our objective of calculating the limits of multivariable functions.

Our strategy will be to combine the results of Theorems 1a and 1b to simplify a limit of a complicated function:

\[
\lim_{(x,y) \to (1,-2)} (y^2 + \sqrt{-xy})
\]

\[
= \lim_{(x,y) \to (1,-2)} y^2 + \lim_{(x,y) \to (1,-2)} \sqrt{-xy}
\]

\[
= (\lim_{(x,y) \to (1,-2)} y)^2 + \sqrt{\lim_{(x,y) \to (1,-2)} -xy}
\]

\[
= (-2)^2 + \sqrt{-\left(\lim_{(x,y) \to (1,-2)} xy\right)}
\]

\[
= 4 + \sqrt{-\left(\lim_{(x,y) \to (1,-2)} x\right)\left(\lim_{(x,y) \to (1,-2)} y\right)}
\]

\[
= 4 + \sqrt{-1(-2)}
\]

\[
= 4 + \sqrt{2}
\]
Example: Use Theorems 1a and 1b to calculate the following limit (show all of your steps)

$$\lim_{(x,y) \to (-3,2)} \frac{3xy^2}{x + 2y^3}.$$ 

Th. 1b

$$= 3 \left( \lim_{(x,y) \to (-3,2)} x \right) \left( \lim_{(x,y) \to (3,2)} y \right)^2$$

$$= \frac{\left( \lim_{(x,y) \to (-3,2)} x \right) + 2 \left( \lim_{(x,y) \to (-3,2)} y \right)^3}{\left( \lim_{(x,y) \to (-3,2)} x \right) + 2 \left( \lim_{(x,y) \to (-3,2)} y \right)^3}$$

by Th. 1(a)

$$= \frac{3 (-3) \left( \frac{2}{13} \right)^2}{(-3) + 2 \left( \frac{2}{13} \right)^3} = -\frac{36}{13}$$
While we have learned a great deal in our study of limits, there are still some unanswered questions.

For example, how would you calculate the following

\[ \lim_{(x,y) \to (-3,2)} \sin(x^2 + y^2) ? \]

To answer this question will take further developments with a slight detour along the way.

**Limits at Boundary Points**

Let \( R \) be a region in \( \mathbb{R}^2 \). An **interior point** \( P \) of \( R \) lies entirely within \( R \), which means it is possible to find a disk centered at \( P \) which contains only points of \( R \).

A **boundary point** \( Q \) of \( R \) lies on the edge of \( R \) in the sense that every disk which contains \( Q \) has a point inside of \( R \) and a point outside of \( R \).
The utilization of Theorems 1a and 1b may not be enough to calculate the limit of \( f(x, y) \) at a boundary point for the domain of the function.

Example: Calculate the following limit

\[
\lim_{(x,y) \to (1,1)} \frac{x^2 - y^2}{x - y}.
\]

\[
= \lim_{(x,y) \to (1,1)} \frac{(x+y)(x-y)}{(x-y)} = \frac{1+1}{1} = 2.
\]

Example: Calculate the following limit

\[
\lim_{(x,y) \to (1,2)} \frac{\sqrt{y} - \sqrt{x+1}}{y - x - 1}.
\]

\[
= \lim_{(x,y) \to (1,2)} \frac{(\sqrt{y} - \sqrt{x+1})(\sqrt{y} + \sqrt{x+1})}{(y-x-1)(\sqrt{y} + \sqrt{x+1})}
\]

\[
= \lim_{(x,y) \to (1,2)} \frac{y - x - 1}{(y-x-1)(\sqrt{y} + \sqrt{x+1})}
\]

\[
= \frac{1}{\sqrt{2} + \sqrt{2}} = \frac{1}{2\sqrt{2}}
\]
Continuity of Multivariable Functions

For a real-valued function of a single variable \( f(x) \), we know that the function is **continuous** at \( x = a \) if \( f(x) \) is defined at \( x = a \), \( \lim_{x \to a} f(x) \) exists, and \( \lim_{x \to a} f(x) = f(a) \).

A similar result holds for multivariable functions:

A function \( f(x, y) \) is continuous at a point \((x, y) = (a, b)\) if

\[
\begin{align*}
\text{a. } f(x, y) \text{ is defined at } (x, y) = (a, b) \quad &\text{ and } \quad \left[ \lim_{(x, y) \to (a, b)} f(x, y) \text{ exists} \right] \\
\text{b. } \lim_{(x, y) \to (a, b)} f(x, y) \quad &\text{ exist} \\
\text{c. } \lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)
\end{align*}
\]

The concept of continuity is **the key** for calculating the limits of multivariable functions. If a function is continuous at a point, the limit exists and **is equal** to the value of the function at a point.

Using Theorems 1a and 1b, we can prove that multivariable polynomials are continuous at all points and multivariable rational functions are continuous at all points in their domain. Thus calculating the limits of these functions is quite easy.
Example: Calculate the following limit

\[
\lim_{(x,y) \to (1,2)} \underbrace{x^2 - 3xy + y^3 - 2}_{\text{polynomial}} = 1.
\]

Example: Calculate the following limit

\[
\lim_{(x,y) \to (1,2)} \frac{y - x^3}{y + x + 1}.
\]

\[
\lim_{(x,y) \to (1,1)} \frac{2 - 1^3}{2 + 1 + 1} = \frac{1}{4}.
\]
We are almost to the end!

It can be proved that the basic properties of continuous functions of a single variable carry over to multivariable functions; that is the sum, difference, product, composition, and quotients of continuous functions are continuous (assuming there is no division by zero in the case of quotients).

To illustrate the importance of these properties in the determination of limits recall the following question:

\[ \lim_{(x,y) \to (-3,2)} \sin(x^2 + y^2) = ? \]

a. The function \( x^2 + y^2 \) is continuous at \((-3, 2)\) (why?); polynomial

b. \( \sin u \) is continuous at \( u = 13 \) (why?); \( \sin(u) \) is a continuous function.

c. \( \sin(x^2 + y^2) \) is continuous at \((-3, 2)\) (why?); composition of \( \sin(\cdot) \) & \( x^2 + y^2 \)

d. \( \lim_{(x,y) \to (-3,2)} \sin(x^2 + y^2) = \sin 13 \) (why?). continuity
Example: Calculate the following limit

\[
\lim_{(x,y) \to (1, -2)} \frac{e^{x^2+y}}{\sqrt{x^2 + 3 - 2xy}}.
\]

plug in see.

\[
\lim_{(x,y) \to (1, -2)} \frac{e^{1^2-2}}{\sqrt{1^2+3 - 2(1)(-2)}} = \frac{e^{-1}}{2+4} = \frac{1}{6e}.
\]
Final Thoughts

Throughout most of this lecture we have focused our discussion on functions of two variables. The ideas and techniques follow through naturally for functions of three or more variables by making minor yet significant changes as needed.

Note that the limit definition can be extended to a function of three variables as follows:

Definition: The function \( g(x, y, z) \) has the limit \( L \) as \( (x, y, z) \to (a, b, c) \), written

\[
\lim_{(x,y,z)\to(a,b,c)} g(x, y, z) = L
\]

if, given any \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that

\[
|g(x, y, z) - L| < \epsilon
\]

whenever

\[
0 < \sqrt{(x - a)^2 + (y - b)^2 + (z - c)^2} < \delta.
\]

We will allow the student to reflect upon the appropriate changes in the remaining ideas developed in this lecture.