The Chain Rule with One Independent Variable

Recall that in Calculus I we introduced the chain rule to simplify the process of differentiating composite functions. Given the function \( y = f(g(x)) \), the chain rule can be written as

\[
\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \quad \text{or} \quad y' = f'(g(x)) \cdot g'(x).
\]

Let's modify our notation slightly and revisit this idea.

Suppose we start with a function \( z = f(x) \) where the variable \( x \) is itself a function of a variable \( t \), \( x(t) \). In this case we write \( z = f(x(t)) \).

We will refer to the variable \( z \) as the dependent variable, the variable \( x \) as the intermediate variable, and the variable \( t \) as the independent variable.

In this case the chain rule is written as

\[
\frac{dz}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt}.
\]
Let us now extend this concept to a function $f$ with two intermediate variables, $x$ and $y$, and a single independent variable $t$:

$$z = f(x, y) = f(x(t), y(t)).$$

Note that the derivative $dz/dt$ is a standard derivative since we are determining the derivative of $z$ with respect to the single independent variable $t$.

The chain rule for this case is given by the following:

**The Chain Rule with One Independent Variable**

Let $z$ be a differentiable function of $x$ and $y$ on its domain where $x$ and $y$ are differentiable functions of $t$ on $I$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

How do you explain the mix of standard and partial derivatives in this version of the chain rule?

$f$ is a function of two intermediate variables, hence $\partial$

whereas $x$ and $y$ are functions of a single independent variable $t$, hence $d$
Example: Compute $dz/dt$ where $z = x^2 y + e^x$, $x = 2t$, and $y = 3t^2$.

\[
\begin{align*}
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\
&= ((2xy + e^x) \cdot (2) + (x^2) \cdot (6t)) \\
\frac{dz}{dt} &= [2(2t)(3t^2) + e^{2t}(3)(2)] + (2t)^2 (6t) \\
\frac{dz}{dt} &= 48t^3 + 2e^{2t} \]
\end{align*}
\]

Example: Compute $dz/dt$ where $z = \sqrt{x + y}$, $x = \cos 2t$, and $y = \sin 2t$.

\[
\begin{align*}
\frac{dz}{dt} &= \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \\
&= \left(\frac{1}{2\sqrt{x+y}}\right) (-2\sin 2t) + \frac{1}{2\sqrt{x+y}} (2\cos 2t) \\
\frac{dz}{dt} &= \frac{\cos(2t) - \sin(2t)}{\sqrt{\cos(2t) + \sin(2t)}}
\end{align*}
\]
Similar formulas for the chain rule can be developed if there are more than two intermediate variables yet still only one independent variable. For example, if

\[ w = g(x, y, z) = g(x(t), y(t), z(t)) \]

then

\[ \frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}. \]

As a way to remember the correct form of the chain rule, it is often useful to construct a tree diagram:

Example: Compute \( \frac{dw}{dt} \) where \( w = yze^x, x = t, y = e^t, \) and \( z = \ln t. \)

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt} = (yze^x)(1) + z e^x(e^t) + Ye^x(\frac{1}{t})
\]

\[
= (e^t(\ln t) e^t) + (\ln t . e^t) e^t + e^t e^t / t
\]

\[
= e^{2t} (2 \ln(t) + 1/t)
\]
The Chain Rule with Two Independent Variables

Suppose now we have a function of the form \( z = f(x, y) \) where both \( x \) and \( y \) are functions of the variables \( s \) and \( t \).

In this case the dependent variable is \( z \), the intermediate variables are \( x \) and \( y \), and the independent variables are \( s \) and \( t \).

And we can write \( z = f(x(s,t), y(s,t)) \).

The chain rule for this case is given by:

**Chain Rule (Two Independent Variables)**

Let \( z \) be a differentiable function of \( x \) and \( y \), where \( x \) and \( y \) are differentiable functions of \( s \) and \( t \), then

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}
\]

Why are the derivatives of \( z \) with respect to \( s \) and \( t \) partial derivatives?

\( z \) is a \( \frac{\partial}{\partial z} \) of two variables \( x \) and \( y \).

\( x \) and \( y \) are \( \frac{\partial}{\partial x} \) of two variables \( s \) and \( t \).
What is the tree diagram for this new case?

Example: For $z = x \ln y$, $x = st$, $y = s + t$, calculate the partial derivatives of $z$ with respect to $s$ and $t$.

\[ \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (\ln y)(t) + (x/y)(1) \]

\[ \frac{\partial z}{\partial s} = t \ln(s+t) + \frac{st}{s+t} \]

\[ \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (\ln y)(s) + (x/y) \]

\[ \frac{\partial z}{\partial t} = s \ln(s+t) + \frac{st}{s+t} \]
Example: For $z = 3x^2y^3$, $x = \cos t + \sin s$, $y = \sin t - \cos s$, calculate the partial derivatives of $z$ with respect to $s$ and $t$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial s} = (\cos(x)y^3)(\cos(s)) + (9x^2y^2)(\sin(s))$$

$$\frac{\partial z}{\partial s} = 6(\cos(t)+\sin(s))(\sin(t)-\cos(s))^3(\cos(s))$$

$$+ 9(\cos t+\sin s)^4(\sin t - \cos(s))^2 \sin(s)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial z}{\partial t} = \cos x y^3(\sin t) + (9x^2y^2)\cos t$$

$$\frac{\partial z}{\partial t} = 6(\cos t + \sin(s))(\sin t - \cos s)^3 \sin t + 9(\cos t + \sin s)^2(\sin t - \cos s)^2 \cos t$$

Now consider the case where we have a function $w$ of three intermediate variables $x$, $y$, and $z$, each of which is a function of the two independent variables $s$ and $t$.

What is the tree diagram for this case? What does the chain rule look like?

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}$$
Example: For \( w = x \cos z + y \sin z, \) \( x = s + t, \) \( y = s + t^2, \) and \( z = s^2 + t \) calculate the partial derivatives of \( w \) with respect to \( s \) and \( t. \)

\[
\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial s}
\]

\[
= (\cos z)(1) + (\sin z) 1 + (-x \sin z + y \cos z)(2s)
\]

\[
\frac{\partial w}{\partial s} = [1 + 2s (s+t^2)] \cos(s^2+t) + [-2s(s+t)] \sin (s^2+t)
\]

\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial t}
\]

\[
\frac{\partial w}{\partial t} = (\cos z)(1) + (\sin z)(2t) + (-x \sin z + y \cos z)(1)
\]

\[
\frac{\partial w}{\partial t} = [1 + (s+t^2)] \cos(s^2+t) + [-s^2 - 2t] \sin (s^2+t)
\]
Application

The density of a thin circular plate of radius 2 is given by $\rho(x, y) = 4 + xy$ and the edge of the plate is described by the parametric equations $x = 2 \cos t$, $y = 2 \sin t$, for $0 \leq t \leq 2\pi$.

a. Find the rate of change of density with respect to $t$ on the edge of the plate.

\[
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial x} \frac{dx}{dt} + \frac{\partial \rho}{\partial y} \frac{dy}{dt}
\]

\[
= y(-2\sin(t)) + (x)(2\cos(t))
\]

\[
\frac{d\rho}{dt} = (2\sin t)(-2\sin t) + (2\cos t)(2\cos t)
\]

\[
\frac{d\rho}{dt} = 4(\cos^2 t - \sin^2 t) = 4\cos(2t).
\]

Critical points: $t = \pi/4, 5\pi/4, 3\pi/4, 7\pi/4$.

$\rho(0) = 4$ $\rho(\pi/4) = 6$

$\rho(\pi/4) = 6$ $\rho(3\pi/4) = 2$

$\rho(3\pi/4) = 2$ $\rho(2\pi) = 4$. 

max $C$ $t = \pi/4, 5\pi/4$. 
Implicit Differentiation

Consider a function of the form $F(x, y) = 0$ where $x$ is the independent variable and $y$ is the dependent variable. In a previous lecture we stated that the function $F(x, y)$ is referred to as an implicit function since the dependent variable is not given explicitly as a function of the independent variable.

Using the topics introduced in this lecture we wish to develop a formula for $dy/dx$ for functions given implicitly.

We begin by writing $z = F(x, y) = 0$ and consider $x$ and $y$ to be intermediate variables and $x$ to be the independent variable.

This makes sense since $y$ is the dependent variable and hence is a function of $x$, $y = y(x)$, and $x$ is also a function of itself since

$$x = x$$

Using the chain rule for a single independent variable we obtain

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} \frac{dx}{dx} + \frac{\partial z}{\partial y} \frac{dy}{dx}$$

Since $dx/dx = 1$

and since $z = F(x, y) = 0$ implies $dz/dx = 0$

$z$ is a constant
we obtain \[ 0 = \frac{dz}{dx} + \frac{\partial z}{\partial y} \frac{dy}{dx} \]

which can be rewritten as

\[ \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y} \]

The result is summarized in the following:

**Theorem**

**Implicit Differentiation**

Let \( F \) be differentiable on its domain and suppose \( F(x, y) = 0 \) defines \( y \) as a differentiable function of \( x \). Provided \( F_y \neq 0 \),

\[ \frac{dy}{dx} = -\frac{F_x}{F_y} \]

Example: Given \( \sqrt{x^2 + xy + y^2} = 3 \), calculate \( dy/dx \).

\[ F(x, y) = \sqrt{x^2 + xy + y^2} - 3 = 0 \]
\[ F_x = \frac{1}{2} \left( \frac{1}{\sqrt{x^2 + xy + y^2}} \right) (2x+y) \]
\[ F_y = \frac{1}{2} \left( \frac{1}{\sqrt{x^2 + xy + y^2}} \right) (2y + x) \]

\[ \frac{dy}{dx} = \frac{-(2x+y)}{(2y+x)} \]
The Gradient and Level Curves

Consider the function \( f(x, y) \); if we set this function equal to a constant \( z_0 \), the resulting equality generates a curve in the \( x, y \)-plane called a **level curve**.

Next suppose we parametrize the level curve \( f(x, y) = z_0 \) by a function of the form \( \mathbf{r}(t) = (x(t), y(t)) \), \( t \in [\alpha, \beta] \). We can now express the level curve as \( f(x(t), y(t)) = z_0 \).

Differentiating both sides of the equality with respect to \( t \) and using the chain rule yields

\[
\frac{d}{dt} f(x(t), y(t)) = \frac{d}{dt} (z_0)
\]

\[
\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = 0
\]

\[
\left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \cdot \left( \frac{dx}{dt}, \frac{dy}{dt} \right) = 0
\]

\[
\nabla f \cdot \vec{r}'(t) = 0
\]

This last equality suggests that the gradient vector is orthogonal to the tangent line at each point on the curve.

*since \( \vec{r}'(t) \) is a vector tangent to the curve.*
We have motivated the following theorem:

**Theorem**

**The Gradient and Level Curves**

Given a function $f$ differentiable at $(a, b)$, the line tangent to the level curve of $f$ at $(a, b)$ is orthogonal to the gradient $\nabla f(a, b)$ provided $\nabla f(a, b) \neq 0$.

Example: Given the paraboloid $f(x, y) = 1 - x^2/4 - y^2/16$, verify the results of the theorem when $f(x, y) = 0$ and $(a, b) = (0, 4)$.

When $f(x, y) = 0$ \iff $1 = x^2/4 + y^2/4$ is the level curve

Parametrize the level curve by:

$r(t) = \langle 2\cos t, 4\sin t \rangle \quad t \in [0, 2\pi]$

$r'(t) = \langle -2\sin t, 4\cos t \rangle \quad t \in [0, 2\pi]$

For the point $(0, 4)$ \quad $t = \pi/2$

$r'(\pi/2) = \langle -2, 0 \rangle$. Tangent vector to the level curve at \pi/2

$\nabla f = \langle -x/2, -y/8 \rangle$ \quad $\nabla f(0, 4) = \langle 0, -1/2 \rangle$

$\nabla f(0, 4) \cdot r'(\pi/2) = \langle 0, -1/2 \rangle \cdot \langle -2, 0 \rangle = 0$

Theorem is verified
Consider the function \( F(x, y, z) \); if we set this function equal to a constant \( K \), the resulting equality generates a surface in the \( R^3 \) called a **level surface**. Employing a motivational argument similar to that used with level curves we can derive an analog to the last theorem; that is if \( (a, b, c) \) is a point on the level surface \( F(x, y, z) = K \) and \( \nabla F(a, b, c) \neq \mathbf{0} \), then \( \nabla F(a, b, c) \) is orthogonal to the tangent plane to the level surface at \( (a, b, c) \).

This result is useful in determining the equations of tangent planes to level surfaces.

Example: Find the equation of the tangent plane to the ellipsoid \( x^2 + y^2/4 + z^2/9 = 25 \) at the point \( (3, 8, 0) \).

\[
\begin{align*}
F(x, y, z) &= x^2 + \frac{y^2}{4} + \frac{z^2}{9} - 25 = 0 \\
\nabla F &= \langle 2x, \frac{y}{2}, \frac{2z}{9} \rangle \\
\nabla F \mid_{(3, 8, 0)} &= \langle 6, 4, 0 \rangle
\end{align*}
\]

This is a normal vector for the tangent plane.

The equation of plane \( \langle x-3, y-8, z-0 \rangle \cdot \langle 6, 4, 0 \rangle = 0 \)