Local Maximum, Minimum Values

As we have seen throughout this semester, the concepts of Calculus I - limits, continuity, differentiability - are currently being reexamined in the context of functions of several variables. We continue that trend with the current lecture and discuss max and min values for multivariable functions.

Definition. A function $f(x, y)$ has a **local maximum value** at $(a, b)$ if $f(x, y) \leq f(a, b)$ for all $(x, y,)$ in the domain of $f$ in some open disk centered at $(a, b)$. A function $f$ has a **local minimum value** at $(a, b)$ if $f(x, y) \geq f(a, b)$ for all $(x, y,)$ in the domain of $f$ in some open disk centered at $(a, b)$. Local maximum and local minimum values are also called **local extreme values** or **local extrema**.

How do these definitions differ from those for a real-valued function of a single variable?

Now that we know what local extrema are, how do we find the points at which they occur?
Theorem 1

**Derivatives and Local Max, Min Values**

If \( f \) has a local max or min value at \((a, b)\) and \( f_x \) and \( f_y \) exist at \((a, b)\), then \( f_x(a, b) = f_y(a, b) = 0 \).

The logic behind this theorem is evident if one examines the tangent plane to a surface at a point where the surface has a local max or min.

At this point the tangent plane is \textbf{horizontal}.

Recall that the equation of the tangent plane at \((a, b)\) is given by

\[
z = f(a, b) + f_x(a, b)(x-a) + f_y(a, b)(y-b).
\]

A horizontal plane has an equation of the form \( z = c \), which would imply

\[
f_x(a, b) = f_y(a, b) = 0
\]

What is this theorem \textbf{not} saying?

\( \text{It does not say that if } f_x(a, b) = 0 \text{ and } f_y(a, b) = 0 \text{ then } (a, b) \text{ is a max or min at } (a, b) \).
Definition: An interior point \((a, b)\) in the domain of \(f\) is a **critical point** of \(f\) if either

1. \(f_x(a, b) = f_y(a, b) = 0\), or

2. one (or both) of \(f_x\) or \(f_y\) does not exist at \((a, b)\).

Example: Find all of the critical points for the function \(f(x, y) = x^4 - 2x^2 + y^2 - 4y + 5\).

\[
\begin{align*}
f_x &= 4x^3 - 4x = 4x(x^2 - 1) = 0 \quad \Rightarrow \quad x = 0, 1, -1 \\
f_y &= 2y - 4y = 0 \quad \Rightarrow \quad y = 2.
\end{align*}
\]

**critical points** \((0, 2)\); \((1, 2)\); \((-1, 2)\)

Example: Find all of the critical points for the function \(f(x, y) = 3x^2 - 4y^2\).

\[
\begin{align*}
f_x &= 6x = 0 \quad \Rightarrow \quad x = 0 \quad \text{\( (0, 0)\)} \\
f_y &= -8y = 0 \quad \Rightarrow \quad y = 0
\end{align*}
\]
As we know from our experience in Calculus I, not all critical points correspond to local max and mins for a function.

Consider the function from the last example, 
\[ f(x, y) = 3x^2 - 4y^2. \]

If we intersect the graph of this function with the plane \( y = 0 \), we generate the curve

\[ z = 3x^2 \]

which would lead us to suspect that at the critical point \((0, 0)\) there exists a \textit{local min}.

If we intersect the graph of this function with the plane \( x = 0 \), we generate the curve

\[ z = -4y^2 \]

which would lead us to suspect that at the critical point \((0, 0)\) there exists a

A point such as this is referred to as a \textbf{saddle point}. 
Definition. A function $f$ has a **saddle point** at a critical point $(a, b)$ if, in every open disk centered at $(a, b)$, there are points $(x, y)$ for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

Recall: 2nd derivative test: If $f'(a) = 0$ and (i) $f''(a) < 0 \Rightarrow$ loc. max @ $a$  
(ii) $f''(a) > 0 \Rightarrow$ loc. min @ $a$.

Theorem 2

**Second Derivative Test**

Suppose that the second partial derivatives of $f$ are continuous throughout an open disk centered at the point $(a, b)$, where $f_x(a, b) = f_y(a, b) = 0$. Define a new function called the **discriminant** by $D(x, y) = f_{xx}(x, y)f_{yy}(x, y) - (f_{xy}(x, y))^2$.

1. If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a local max value at $(a, b)$.

2. If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a local min value at $(a, b)$.

3. If $D(a, b) < 0$, then $f$ has a saddle point at $(a, b)$.

4. If $D(x, y) = 0$, then the test is inconclusive.
Example: Find and classify all critical points for 
\[ f(x, y) = 2x^2 + 4y^2 - 5. \]

\[ f_x = 4x = 0 \implies x = 0 \]
\[ f_y = 8y = 0 \implies y = 0 \]

Critical point \((0,0)\)

\[ f_{xx} = 4 \quad ; \quad f_{yy} = 8 \quad ; \quad f_{xy} = 0 \]

\[ D(0,0) = (4)(8) - 32 = 32 \]

\[ D(0,0) > 0 \quad \text{and} \quad f_{xx}(0,0) > 0 \]

\implies \text{local minimum at } (0,0). \text{ Value of } \min_{f(0,0)} = -5.

Example: Find and classify all critical points for 
\[ f(x, y) = e^{4y-x^2-y^2}. \]

\[ f_x = -2x e^{4y-x^2-y^2} \implies x = 0 \]
\[ f_y = (2y+4) e^{4y-x^2-y^2} \implies y = 2 \]

C.P. \((0,2)\)

\[ f_{xx} = -2 e^{4y-x^2-y^2} + 4x^2 e^{4y-x^2-y^2} = (4x^2 + 2) e^{4y-x^2-y^2} \]
\[ f_{yy} = -2 e^{4y-x^2-y^2} + (2y+4)(-2y+4) e^{4y-x^2-y^2} = (-8y^2 + 16y + 16) e^{4y-x^2-y^2} \]
\[ f_{xy} = -2x(2y+4) e^{4y-x^2-y^2} \]

\[ D(0,2) = 4 e^{8} > 0 \quad \text{and} \quad f_{xx}(0,0) = -2 e^{4} < 0 \]

\implies \text{local max at } (0,2). \text{ Value of } \max_{f(0,y) = e^{4y}}
Example: Find and classify all critical points for
\[ f(x, y) = x^3 y + 12x^2 - 8y. \]

\[ f_x = 3x^2 y + 24x = 3x(x^2y + 8) = 0 \quad \text{and} \quad 6(2y + 8) = 0 \]

\[ f_y = x^3 - 8 = 0 \quad \Rightarrow \quad x = 2 \quad \text{and} \quad y = -4 \]

\( C.P. \quad (2, -4) \)

\[ f_{xx} = 6xy + 24 \quad f_{yy} = 0 \quad f_{xy} = 3x^2 \]

\( D(2, -4) = -144 < 0 \quad \text{Saddle point at} \quad (2, -4) \)

Example: Given \( f(x, y) = x^4 + 2y^2 - 4xy \), determine if the function has any local mins and determine the values of the local mins if they exist.

\[ f_x = 4x^3 - 4y = 4(x^3 - y) = 0 \]

\[ f_y = 4y - 4x = 4(y - x) = 0 \quad \Rightarrow \quad x = y \]

\( C.P. \quad (0, 0), (1, 1), (-1, -1) \)

\[ f_{xx} = 12x^2 \quad f_{yy} = 4 \quad f_{xy} = -4 \]

\( D(0, 0) = -16 < 0 \quad \text{Saddle point} \)

\( D(1, 1) = 32 > 0 \quad \text{local min} \quad f_{xx}(1, 1) = 12 > 0 \)

\( D(-1, -1) = 32 > 0 \quad f_{xx}(-1, -1) = 12 > 0 \quad \text{local min} \)
Absolute Max and Min Values

In addition to relative extrema, we also have **global** or **absolute extrema**.

Definition. If \( f(x, y) \leq f(a, b) \) for all \((x, y)\) in the domain of \(f\) then, then \(f\) has an **absolute maximum** at \((a, b)\). If \(f(x, y) \geq f(a, b)\) for all \((x, y)\) in the domain of \(f\) then, then \(f\) has an **absolute minimum** at \((a, b)\).

Recall from Calculus I that a function may have no absolute extrema, one absolute extrema, or two absolute extrema.

Example: Find any absolute extrema of the function \(f(x, y) = 10 - 2x^2 - 5y^2\).
The procedure used to determine the absolute max and min of a differentiable function \( f(x, y) \) on a \textbf{closed} and \textbf{bounded} region is similar to but not exactly the same as that used for a real-valued function of a single variable (what was this procedure?):

1. Determine the value of \( f \) at all critical points in the interior of the closed and bounded region.

2. Find all extrema of \( f \) on the boundary.

3. The greatest function value from steps 1 and 2 is the absolute max, the least function value from steps 1 and 2 is the absolute min.

Example: Find the absolute extrema of \( f(x, y) = x^2 + y^2 - 4 \) on the disk \( x^2 + y^2 \leq 4 \).

(1) \[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x = 0 \implies x = 0 \\
\frac{\partial f}{\partial y} &= 2y = 0 \implies y = 0.
\end{align*}
\]

**Critical Point** \((0, 0)\) \quad \text{Value} \quad f(0, 0) = -4

(2) Parametrize the boundary curve \( x = 2 \cos t \quad y = 2 \sin t \quad t \in [0, 2\pi] \)

on the boundary \( f(x, y) = (2 \cos t)^2 + (2 \sin t)^2 - 4 = 0 \)

\( f(x, y) \) const. on the boundary; no extrema

(3) From (1) \& (2) \quad \text{abs max} \quad 0 \quad \text{abs min} \quad -4.
Example: Find the absolute extrema of \( f(x, y) = 3 + xy - x - 2y \) in the triangular region with vertices (1, 0), (5, 0), and (1, 4).

1. \( f_x = y - 1 = 0 \Rightarrow y = 1 \)
   \( f_y = x - 2 = 0 \Rightarrow x = 2 \)
   \( f(2, 1) = 1 \).

2. (I) boundary curve \( x = 1, \ y \in [0, 4] \).
   \( f(x, y) = f(y, y) = 2 - y \) on \( [0, 4] \).
   \( \max \) at \( y = 4 \) \( f(0, 4) = 2 \)
   \( \min \) at \( y = 0 \) \( f(0, 0) = -2 \)

3. (II) boundary curve \( y = 0, \ x \in [1, 5] \)
   \( f(x, y) = f(x, 0) = 3 - x \)
   \( \max \) at \( x = 1 \) \( f(1, 0) = 2 \)
   \( \min \) at \( x = 5 \) \( f(5, 0) = -2 \)

4. (III) boundary curve \( y = -x + 5, \ x \in [1, 6] \)
   \( f(x, y) = f(x, -x + 5) = -x^2 + 6x + 7 \)
   \( f'(x, -x + 5) = -2x + 6 \Rightarrow x = 3, \)
   \( f(3, 2) = 3 + 6 - 2 - 6 = 1 \)
   \( f(1, 4) = -2 \quad f(5) = -2 \).

From all these values, \( \max = 2 \)
\( \min = -2 \).
Example: Find the point on the surface \( z^2 = xy + 1 \) closest to the origin.

We are trying to minimize \( d = \sqrt{x^2 + y^2 + z^2} \)

\[
d^2 = D = x^2 + y^2 + xy + 1 \quad \Rightarrow \quad d = \sqrt{x^2 + y^2 + xy + 1}
\]

Local min
\[
D_x = 2x + y = 0 \quad x = 0 \quad y = 0
\]
\[
D_y = 2y + x = 0
\]

Critical point
\[
D_{xy} = 2 \quad D_{yy} = 2 \quad D_{xy} = 1.
\]

\[
D(0,0) = 3 > 0 \quad D_{xx} > 0 \quad \therefore \text{local min @ (0,0)}
\]

Is it an abs. min?

Sub \( x = r \cos \theta \) \( y = r \sin \theta \) \( \Rightarrow \)

\[
d^2 = \frac{1}{2} r^2 (2 + \sin 2\theta) + 1
\]

\( d^2 = 1 \) only when \( r = 0 \)

\( d^2 = 1 \) is abs. min.

and occurs at \( (0,0) \).