Cross Product

In this lecture we again define a new product for vectors. In this case, the multiplication results in a product vector with some very interesting properties and applications.

The cross product of the vectors \( \mathbf{v} = (a_1, b_1, c_1) \) and \( \mathbf{w} = (a_2, b_2, c_2) \), represented symbolically as \( \mathbf{v} \times \mathbf{w} \), is defined by

\[
\mathbf{v} \times \mathbf{w} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
a_1 & b_1 & c_1 \\
a_2 & b_2 & c_2
\end{vmatrix}
\]

where the right side of the equality is the determinant of a 3 \( \times \) 3 matrix.

The determinant of the matrix can be computed in many ways; perhaps the simplest technique is to write the three columns of the matrix and add two additional columns by repeating the first and second columns. Next proceed from left to right by multiplying the diagonal elements and summing the products; finally, proceeding from right to left multiply the reverse diagonal elements and subtract these quantities from the previous sum.
\[ + \text{ in this direction } \downarrow \]
\[ - \text{ in this direction } \uparrow \]

\[
\begin{array}{c}
\hat{i} \hat{j} \hat{k} \\
\hat{a}_1 \hat{b}_1 \hat{c}_1 \\
\hat{a}_2 \hat{b}_2 \hat{c}_2 \\
\end{array}
\]

\[
= \hat{b}_1 \hat{c}_2 \hat{i} + \hat{a}_2 \hat{c}_1 \hat{j} + \hat{a}_1 \hat{b}_2 \hat{k} - \hat{a}_2 \hat{c}_2 \hat{i} - \hat{b}_1 \hat{c}_1 \hat{j} - \hat{a}_1 \hat{b}_2 \hat{k} - \hat{a}_2 \hat{b}_1 \hat{i} \hat{k}
\]

\[
= (\hat{b}_1 \hat{c}_2 - \hat{b}_2 \hat{c}_1) \hat{i} + (\hat{a}_2 \hat{c}_1 - \hat{a}_1 \hat{c}_2) \hat{j} + (\hat{a}_1 \hat{b}_2 - \hat{a}_2 \hat{b}_1) \hat{k}
\]

\[ \text{method 2 (preferred)} \]

\[
\begin{array}{c}
\hat{i} \hat{j} \hat{k} \\
\hat{a}_1 \hat{b}_1 \hat{c}_1 \\
\hat{a}_2 \hat{b}_2 \hat{c}_2 \\
\end{array}
\]

\[
= \hat{b}_1 \hat{c}_1 \hat{i} - \hat{a}_1 \hat{c}_1 \hat{j} + \hat{a}_1 \hat{b}_1 \hat{k}
\]

\[
= \hat{b}_1 \hat{c}_2 \hat{i} - \hat{a}_2 \hat{c}_1 \hat{j} + \hat{a}_2 \hat{b}_1 \hat{k}
\]

\[
= \hat{b}_1 \hat{c}_2 \hat{i} - \hat{a}_2 \hat{c}_1 \hat{j} + \hat{a}_2 \hat{b}_1 \hat{k}
\]

\[
= (\hat{b}_1 \hat{c}_2 - \hat{b}_2 \hat{c}_1) \hat{i} - (\hat{a}_2 \hat{c}_1 - \hat{a}_1 \hat{c}_2) \hat{j} + (\hat{a}_1 \hat{b}_2 - \hat{a}_2 \hat{b}_1) \hat{k}
\]

\[ \text{Example: Calculate the cross product } \vec{u} \times \vec{v} \text{ where } \]
\[ \vec{u} = \langle 1, 3, -2 \rangle \text{ and } \vec{v} = \langle -3, 5, 4 \rangle. \]

\[
\hat{u} \times \hat{v} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 3 & -2 \\
-3 & 5 & 4 \\
\end{vmatrix}
\]

\[
= \hat{i} \begin{vmatrix} 3 & -2 \\ 5 & 4 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & -2 \\ -3 & 4 \end{vmatrix} + \hat{k} \begin{vmatrix} 1 & 3 \\ -3 & 5 \end{vmatrix}
\]

\[
= \hat{i} (12 + 10) - \hat{j} (4 + 6) + \hat{k} (5 + 9)
\]

\[
= \langle 22, 2, 14 \rangle.
\]
Geometric Description of the Cross Product

We note that the cross product of two nonparallel vectors \( \mathbf{v} \) and \( \mathbf{w} \) results in a third vector \( \mathbf{v} \times \mathbf{w} \) which is easily shown (Theorem 1 below) to be orthogonal to the plane formed by the vectors \( \mathbf{v} \) and \( \mathbf{w} \).

This however leaves two possible orientations of the cross product vector: pointing 'above' the plane or 'below' the plane.

The correct orientation is decided by applying the right hand rule. According to the right hand rule to determine the direction of \( \mathbf{v} \times \mathbf{w} \), you point the outstretched fingers of the right hand in the direction of the vector \( \mathbf{v} \) and curl these fingers in the direction of \( \mathbf{w} \); the direction in which your thumb is pointing during this operation is the direction of \( \mathbf{v} \times \mathbf{w} \).
Example: According to this rule, if your paper corresponds to the $x, y$-plane with the positive $y$-axis pointing to the top of the page and the positive $x$-axis pointing to the right of the page then, $\mathbf{i} \times \mathbf{j}$ is perpendicular to the page pointing upward and $\mathbf{j} \times \mathbf{i}$ is perpendicular to the page pointing downward.

The following theorem summarizes some of the geometric properties of the cross product:

Theorem

The cross product $\mathbf{v} \times \mathbf{w}$ is the unique vector such that.

1. $\mathbf{v} \times \mathbf{w}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$ (and hence the plane formed by these two vectors).

2. $||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| ||\mathbf{w}|| \sin \theta$ where $0 \leq \theta \leq \pi$ is the angle between the two vectors.

How could we easily verify property 1?  

Show $$(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{v} = 0$$

and

$$((\mathbf{v} \times \mathbf{w}) \cdot \mathbf{w} = 0$$

did in class.

When are two vectors parallel? $$(\mathbf{u} \times \mathbf{v})$$

When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$
Example: Find a vector perpendicular to the plane containing the points \((1, 0, 0)\), \((2, 0, -2)\), and \((1, 1, 3)\).

First, form two vectors by subtracting (terminal-start)

\[\overrightarrow{AB} = \langle 1, 0, -2 \rangle \]  \[\overrightarrow{AC} = \langle 0, 1, 3 \rangle \]

Compute \(\overrightarrow{AB} \times \overrightarrow{AC}\) which is perpendicular to the plane.

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 0 & -2 \\
0 & 1 & 3 \\
\end{vmatrix}
= \langle 2, -3, 1 \rangle
\]

**Properties of the Cross Product**

1. \(\overrightarrow{w} \times \overrightarrow{v} = -\overrightarrow{v} \times \overrightarrow{w}\) \textit{anti-commutative.}

2. \(\overrightarrow{v} \times \overrightarrow{v} = \overrightarrow{0}\) why? \(\theta = 0\) and \(\|\overrightarrow{v}\| = \|\overrightarrow{v}\| \sin \theta\)

3. \(\overrightarrow{v} \times \overrightarrow{w} = \overrightarrow{0}\) if and only if \(\overrightarrow{w} = \lambda \overrightarrow{v}\) \textit{parallel} or \(\overrightarrow{v} = \overrightarrow{0}\)

4. \((\lambda \overrightarrow{v}) \times \overrightarrow{w} = \overrightarrow{v} \times (\lambda \overrightarrow{w}) = \lambda (\overrightarrow{v} \times \overrightarrow{w})\)

5. \((\overrightarrow{u} + \overrightarrow{v}) \times \overrightarrow{w} = \overrightarrow{u} \times \overrightarrow{w} + \overrightarrow{v} \times \overrightarrow{w}\) \textit{distributive law. (ORDER MATTERS)}

6. \(\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{w}\)
Application

The torque, or twisting effect, produced by a force $\mathbf{F}$ about a point $O$ when the force is applied at the terminal point of a vector $\overrightarrow{OP}$ is defined by $\tau = \overrightarrow{OP} \times \mathbf{F}$.

Note that the torque vector is perpendicular to the plane formed by $\overrightarrow{OP}$ and $\mathbf{F}$, the maximum torque is produced when the two vectors are orthogonal, and there is no torque when the vectors are parallel.

$$\|\tau\| = \|\overrightarrow{OP}\| \|\mathbf{F}\| \sin \theta$$

Max when $\sin \theta = 1 \Rightarrow \theta = \pi/2$; min $\sin \theta = 0 \Rightarrow \theta = 0$ or $\theta = \pi$

Example: Let $\overrightarrow{OP} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{F} = \langle 20, 0, 0 \rangle$; calculate the magnitude of the torque about $O$.

$$\tau = \overrightarrow{OP} \times \hat{z} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 20 & 0 & 0 \end{vmatrix} = \langle 0, 40, 20 \rangle$$

$$\|\tau\| = \sqrt{(40)^2 + (20)^2} = 20\sqrt{5}$$
Cross Product and Area

There exists an interesting relationship between the cross product and area:

Theorem 3

If the vectors \( \mathbf{u} \) and \( \mathbf{v} \) form two sides of a parallelogram then the area of the parallelogram is

\[
A = \| \mathbf{u} \times \mathbf{v} \|
\]

Example: Given the triangle in the \( x, y \)-plane with vertices \((0, 0)\), \((2, 0)\), and \((0, 3)\), calculate its area using a cross product.

Area of the triangle = \( \frac{1}{2} \) area of parallelogram.

\[
\mathbf{u} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}
\]

\[
\mathbf{u} \times \mathbf{v} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}
\]

area = \( \frac{1}{2} \| \mathbf{u} \times \mathbf{v} \| = 3 \).
Additional Examples

Example: Find all vectors \( \mathbf{u} \) which satisfy the equation

\[
\langle 1, 1, 1 \rangle \times \mathbf{u} = \langle 0, 0, 1 \rangle.
\]

Let \( \mathbf{v} = \langle a, b, c \rangle \)

\[
\langle 1, 1, 1 \rangle \times \langle a, b, c \rangle = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
1 & 1 & 1 \\
a & b & c
\end{vmatrix}
\]

\[
= c \hat{i} + a \hat{j} + b \hat{k} - a \hat{k} - b \hat{i} - c \hat{j}
\]

\[
= \langle c - b, a - c, b - a \rangle = \langle 0, 0, 1 \rangle
\]

\( c = b \) and \( a \neq 1 = b \)

\( a = c \) gives \( a = b \)

which imply no such vector exists.
Example: Determine the direction of the vector $\mathbf{j} \times -\mathbf{k}$.

$$\mathbf{j} \times (-\mathbf{k}) = - (\mathbf{j} \times \mathbf{k}) = - (\mathbf{i}) = -\mathbf{i}$$

Example: Determine the direction of the vector $-\mathbf{j} \times -\mathbf{i}$.

$$(-\mathbf{j}) \times (-\mathbf{i}) = (-1)(-1)(\mathbf{j} \times \mathbf{i}) = \mathbf{j} \times \mathbf{i} = \mathbf{k}$$

Cross product of two of these in the direction of arrow is the other one. In the opposite direction, it is negative.