Functions of Several Variables

Throughout the remainder of the semester we will study the calculus of real-valued functions of the form \( f(x, y) \) and \( g(x, y, z) \), that is functions which contain two or three independent variables and which have output values in the real numbers. As was the case for functions of a single independent variable we will be interested in the domains, ranges, graphs, continuity, differentiability, and integrability of these functions.

We will use the standard definitions of Domains and Ranges for multivariable functions:

The domain is the set of all possible input values for the function.

Note that the domain of a function of the form \( f(x, y) \) is a subset of \( \mathbb{R}^2 \) while the domain of \( g(x, y, z) \) is a subset of \( \mathbb{R}^3 \).

The range of a function is the set of all possible output values; note that all of our functions are real-valued so that range is a subset of \( \mathbb{R} \).
Example: Find the domain and range of the function: 
\( f(x, y) = x^2 + y^2. \)

\[ \begin{align*}
  f(x, y) & \quad \text{is defined for any value of } x \\
  \therefore \text{domain} & = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2
\end{align*} \]

\[ \begin{align*}
  f(x, y) & \quad \text{takes only non negative values} \\
  \therefore \text{range} & = [0, \infty)
\end{align*} \]

Example: Find the domain and range of the function: 
\( f(x, y) = x^2 \sin y. \)

\[ \begin{align*}
  \text{domain} & = \mathbb{R}^2 \\
  \text{fix } y, \quad f(x, c) = x^2 \sin(c) & \quad \to \quad (0, \infty) \; \text{(+)} \\
 & \quad \text{if } \sin(c) > 0 \\
 & \quad \text{(-)} \quad \text{if } \sin(c) < 0
\end{align*} \]

\[ \begin{align*}
  \therefore \text{range} & = [0, \infty) = \mathbb{R}
\end{align*} \]

Example: Find the domain and range of the function: 
\( f(x, y) = x/(y - 3). \)

\[ \begin{align*}
  \text{domain} & \quad y \neq 3, \quad \mathbb{R} \times \left\{ (-\infty, 3) \cup (3, \infty) \right\} \\
  \text{range} & \quad (-\infty, \infty) = \mathbb{R}
\end{align*} \]
Example: Find the domain and range of the function:
\( f(x, y) = \ln(x + y) \).

\[
\begin{align*}
\text{domain} \quad x + y > 0 &= \{(x, y) \in \mathbb{R}^2 \mid x + y > 0\} \\
\text{range} \quad (-\infty, \infty) &= \mathbb{R}
\end{align*}
\]

Example: Find the domain and range of the function:
\( f(x, y) = \sqrt{1 - (x^2 + y^2)} \).

\[
\begin{align*}
\text{domain} &= \left\{(x, y) \in \mathbb{R}^2 \mid 1 - (x^2 + y^2) \geq 0\right\} \\
&= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \\
\text{range} &= [0, 1].
\end{align*}
\]

Example: Find the domain and range of the function:
\( g(x, y, z) = x^2 + y^2 + z^2 \).

\[
\begin{align*}
\text{domain} &= \mathbb{R}^3 \\
\text{range} &= [0, \infty)
\end{align*}
\]
Example: Find the domain and range of the function:
\( g(x, y, z) = 2x + 3y - z \).

\[
\begin{align*}
\text{domain} & \quad = \quad \mathbb{R}^3 \\
\text{range} & \quad = \quad \mathbb{R}
\end{align*}
\]

Example: Find the domain and range of the function:
\( g(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2) \).

\[
\begin{align*}
\text{domain} & \quad 16 - 4x^2 - 4y^2 - z^2 > 0 \quad \Rightarrow \quad 4x^2 + 4y^2 + z^2 < 16 \\
D & = \{ (x, y, z) \in \mathbb{R}^3 \mid 4x^2 + 4y^2 + z^2 < 16 \} \\
\text{range} : & \quad \text{The maximum value the inside can take is 16}, \\
& \quad \text{minimum is 0} \\
\text{range} & \quad = \quad (-\infty, \ln 16]
\end{align*}
\]

To find the domain.

Use the usual restrictions to get an inequality
* denominator \( \neq 0 \)
* Square root, inside \( \geq 0 \)
* Natural log / log, inside \( > 0 \)
* exp, polynomials, no restriction.

To find the range.

1. Make all but one variable constant.
2. See all the values you can get, find the values you cannot obtain by that variable only.
3. Now make that variable a constant, see whether you could get the restricted values from another variable.
Next, let’s explore the two general representations of functions. A real-valued function is in **explicit** form if the dependent variable can be expressed explicitly as a function of the independent variable(s):

\[ y = h(x) \quad z = f(x, y) \quad w = g(x, y, z). \]

A real-valued function is in **implicit** form if it is expressed as a constant equal to a function of all variables both dependent and independent:

\[ 0 = H(x, y) \quad 0 = F(x, y, z) \quad 0 = G(x, y, z, w). \]

For example, explicit functions: the parabola \( y = x^2 \) and the paraboloid \( z = x^2 + y^2 \); implicit functions: the plane \( 2x - 3y - 4z = 5 \) and hemisphere \( 4 = x^2 + y^2 + z^2, \ z \geq 0 \).

Example: Rewrite the implicit function

\[ 4 = x^2 + y^2 + z^2, \ z \geq 0 \]

as an explicit function of the dependent variable \( z \).

\[ z^2 = 4 - (x^2 + y^2) \]

\[ z = \sqrt{4 - (x^2 + y^2)} \quad \text{since} \quad z \geq 0 \]
Example: For the function \( f(x, y) = \sqrt{25 - x^2 - y^2} \), sketch the level curves when \( z_0 = 5, 2\sqrt{6}, \sqrt{21}, 4, 3, 0 \). What is the surface?

\[
f(x, y) = z_0 = 5 = \sqrt{25 - x^2 - y^2} \implies 25 = 25 - x^2 - y^2 \\
\implies x^2 + y^2 = 0 \quad \text{a point}
\]

\( z_0 = 2\sqrt{6} = \sqrt{25 - x^2 - y^2} \implies 24 = 25 - x^2 - y^2 \\
\implies x^2 + y^2 = 1 \quad \text{circle with radius 1}
\]

\( z_0 = \sqrt{21} = \sqrt{25 - x^2 - y^2} \implies 21 = 25 - x^2 - y^2 \\
\implies x^2 + y^2 = 4 \quad \text{circle with radius 2}
\]

\( z_0 = 4 = \sqrt{25 - x^2 - y^2} \implies 16 = 25 - x^2 - y^2 \\
\implies x^2 + y^2 = 9 \quad \text{circle with radius 3}
\]

\( z_0 = 3 = \sqrt{25 - x^2 - y^2} \implies 9 = 25 - x^2 - y^2 \\
\implies x^2 + y^2 = 16 \quad \text{circle with radius 4}
\]

\( z_0 = 0 = \sqrt{25 - x^2 - y^2} \implies 0 = 25 - x^2 - y^2 \\
\implies x^2 + y^2 = 25 \quad \text{circle with radius 5}
\]
Example: For the function \( f(x, y) = 1 + 4x^2 + y^2 \), express the level curves of this function in the form of a standard conic section and sketch the level curves when \( z_0 = 1, 2, 5 \). What is the surface?

The level curves of this function are of the form \( 4x^2 + y^2 = z_0 - 1 \) are ellipses.

- \( z_0 = 1 \)
  - \( 4x^2 + y^2 = 0 \) a point \((0, 0)\)
- \( z_0 = 2 \)
  - \( 4x^2 + y^2 = 2 - 1 \)
  - \( 4x^2 + y^2 = 1 \)
- \( z_0 = 5 \)
  - \( 4x^2 + y^2 = 4 \)
  - \( x^2 + y^2 = 1 \)

Recall, an ellipse has equation \( ax^2 + by^2 = 1 \).

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1
\]
Now consider the graph of a function of three linearly independent variables, \( g(x, y, z) \); the graph of this function would lie in \( \mathbb{R}^4 \) and thus is beyond the visualization of most mortals. Is it possible to gain any qualitative knowledge about this graph?

We can extend the concept of level curves to this case and generate **level surfaces**; if \( g(x, y, z) \) is a real-valued function and \( u_0 \) a constant, then \( u_0 = g(x, y, z) \) is a level surface given in implicit form.

Example: Characterize the level surfaces of the function

\[
g(x, y, z) = 2x + 3y - z.
\]

\[
\nabla W_0 = 2x + 3y - z
\]

planes

Example: Characterize the level surfaces of the function

\[
g(x, y, z) = x^2 + y^2 + z^2.
\]

\[
\nabla W_0 = x^2 + y^2 + z^2 = W_0
\]

surface of spheres
Example: Characterize the level surfaces of the function

\[ g(x, y, z) = \ln(16 - 4x^2 - 4y^2 - z^2). \]

\[ W_0 = \ln \left( 16 - 4x^2 - 4y^2 - z^2 \right) \]

\[ e^{W_0} = 16 - 4x^2 - 4y^2 - z^2 \quad \text{(const)} \]

\[ 4x^2 + 4y^2 + z^2 = \frac{e^{W_0} - 16}{\text{const}} \]

Ellipsoids.