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A CHARACTERIZATION OF HYPERBOLIC AFFINE ITERATED FUNCTION SYSTEMS

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Abstract. The two main theorems of this paper provide a characterization of hyperbolic affine iterated function systems defined on $\mathbb{R}^m$. Atsushi Kameyama [*Distances on topological self-similar sets,* in Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot. Proceedings of Symposia in Pure Mathematics, Volume 72, Part 1, 2004, pages 117–129] asked the following fundamental question: Given a topological self-similar set, does there exist an associated system of contraction mappings? Our theorems imply an affirmative answer to Kameyama's question for self-similar sets derived from affine transformations on $\mathbb{R}^m$.

1. Introduction

The goal of this paper is to prove and explain two theorems that characterize hyperbolic affine iterated function systems defined on $\mathbb{R}^m$. One motivation was the following question: When are the functions of an affine iterated function system (IFS) on $\mathbb{R}^m$ contractions with respect to a metric equivalent to the usual Euclidean metric?

Theorem 1.1 (Characterization for Affine Hyperbolic IFSs)**.** *If $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is an affine iterated function system, then the following statements are equivalent.*

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(1) *$\mathcal{F}$ is hyperbolic.*
(2) *$\mathcal{F}$ is point-fibered.*
(3) *$\mathcal{F}$ has an attractor.*
(4) *$\mathcal{F}$ is a topological contraction with respect to some convex body $K \subset \mathbb{R}^m$.*
(5) *$\mathcal{F}$ is non-antipodal with respect to some convex body $K \subset \mathbb{R}^m$.*

Statement (1) is a metric condition on an affine IFS, statements (2) and (3) are in terms of convergence, and statements (4) and (5) are in terms of concepts from convex geometry. The terms "contractive," "hyperbolic," "point-fibered," "attractor," "topological contraction," and "non-antipodal" are defined in definitions 2.2, 2.3, 2.5, 2.7, 5.8, and 6.5, respectively. This theorem draws together some of the main concepts in the theory of iterated function systems. Banach's classical Contraction Mapping Theorem states that a contraction f on a complete metric space has a fixed point x_0 and that $x_0 = \lim_{k\to\infty} f^{\circ k}(x)$, independent of x, where $\circ k$ denotes the k^{th} iteration. The notion of hyperbolic generalizes the contraction property to an IFS. Namely, an IFS $\mathcal{F}$ is hyperbolic if there is a metric on $\mathbb{R}^m$, Lipschitz equivalent to the usual one, such that each $f \in \mathcal{F}$ is a contraction. The notion of point-fibered, introduced by Bernd Kieninger [10], is the natural generalization of the limit condition above to the case of an IFS. While traditional discussions of fractal geometry focus on the existence of an attractor for a hyperbolic IFS, Theorem 1.1 establishes that the more geometrical (and non-metric) assumptions – topologically contractive and non-antipodal – can also be used to guarantee the existence of an attractor. Basically, a function $f : \mathbb{R}^m \to \mathbb{R}^m$ is non-antipodal if certain pairs of points (antipodal points) on the boundary of K are not mapped by f to another pair of antipodal points.

Since the implication (1) $\Rightarrow$ (2) is the Contraction Mapping Theorem when the IFS contains only one affine mapping, Theorem 1.1 contains an affine IFS version of the converse to the Contraction Mapping Theorem. Thus, our theorem provides a generalization of results proved by Ludvík Janoš [8] and Solomon Leader [12]. Such a converse statement in the IFS setting has remained unclear until now.

Although not every affine IFS $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is hyperbolic on all of $\mathbb{R}^m$, the second main result states that if $\mathcal{F}$ has a coding map (Definition 2.4), then $\mathcal{F}$ is always hyperbolic on some affine subspace of $\mathbb{R}^m$.

Theorem 1.2. *If $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is an affine IFS with a coding map $\pi : \Omega \to \mathbb{R}^m$, then $\mathcal{F}$ is hyperbolic on the affine hull of $\pi(\Omega)$. In particular, if $\pi(\Omega)$ contains a non-empty open subset of $\mathbb{R}^m$, then $\mathcal{F}$ is hyperbolic on $\mathbb{R}^m$.*

Using slightly different terminology, Atsushi Kameyama [9] posed the following fundamental question: *Is an affine IFS with a coding map $\pi : \Omega \to \mathbb{R}^m$ hyperbolic when restricted to $\pi(\Omega)$?* An affirmative answer to this question follows immediately from Theorem 1.2.

Our original motivation, however, was not Kameyama's question, but rather a desire to approximate a compact subset $T \subset \mathbb{R}^m$ as the attractor A of an iterated function system $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ...f_N)$, where each $f_n : \mathbb{R}^m \to \mathbb{R}^m$ is affine. This task is usually done using the "collage theorem" [1], [2] by choosing an IFS $\mathcal{F}$ so that the Hausdorff distance $d_{\mathbb{H}}(T, \mathcal{F}(T))$ is small. If the IFS $\mathcal{F}$ is hyperbolic, then we can guarantee it has an attractor A such that $d_{\mathbb{H}}(T, A)$ is comparably small. But then the question arises: How does one know if $\mathcal{F}$ is hyperbolic?

The paper is organized as follows. Section 2 contains notation, terminology, and definitions that will be used throughout the paper. Section 3 contains examples and remarks relating iterated function systems and their attractors to Theorem 1.1 and Theorem 1.2. In Example 3.1, we show that an affine IFS can be point-fibered, but not contractive under the usual metric on $\mathbb{R}^m$. Thus, some kind of remetrization is required for the system to be contractive. In Example 3.2, we show that an affine IFS can contain two linear maps each with real eigenvalues all with magnitudes less than 1, but still may not be point-fibered. Thus, Theorem 1.1 cannot be phrased only in terms of eigenvalues and eigenvectors of the individual functions in the IFS. Indeed, in Example 3.3, we explain how, given any integer $M > 0$, there exists a linear IFS $\left(\mathbb{R}^2; L_1, L_2\right)$ such that each operator of the form $L_{\sigma_1} L_{\sigma_2} ... L_{\sigma_k}$, with $\sigma_j \in \{1, 2\}$ for $j = 1, 2, ..., k$, and $k \leq M$, has spectral radius less than one, while $L_1 L_2^M$ has spectral radius larger than one. This is related to the joint spectral radius [16] of the pair of linear operators L_1 and L_2 and to the associated finiteness conjecture; see, for example, [4]. In section 8 we comment on the relationship between the present work and recent results concerning the joint spectral radius of finite sets of linear operators. Example 3.4 provides an affine IFS on $\mathbb{R}^2$ that has a coding map π but is not point-fibered on $\mathbb{R}^2$, and hence, by Theorem 1.1, not hyperbolic on $\mathbb{R}^2$. It is, however, point-fibered and hyperbolic when restricted to the x-axis, which is the affine hull of $\pi(\Omega)$, thus illustrating Theorem 1.2.

For the proof of Theorem 1.1, we provide the following road map.

(1) The proof that statement (1) $\Rightarrow$ statement (2) is provided in Theorem 4.1.
(2) The proof that statement (2) $\Rightarrow$ statement (3) is provided in Theorem 4.3.
(3) The proof that statement (3) $\Rightarrow$ statement (4) is provided in Theorem 5.10.
(4) The proof that statement (4) $\Rightarrow$ statement (5) is provided in Proposition 6.6.
(5) The proof that statement (5) $\Rightarrow$ statement (1) is provided in Theorem 6.7.

Theorem 1.2 is proved in section 7.

2. Notation and definitions

We treat $\mathbb{R}^m$ as a vector space, an affine space, and a metric space. We identify a point $x = (x_1, x_2, ..., x_m) \in \mathbb{R}^m$ with the vector whose coordinates are $x_1, x_2, ..., x_m$. We write $0 \in \mathbb{R}^m$ for the point in $\mathbb{R}^m$ whose coordinates are all zero. The standard basis is denoted $\{e_1, e_2, \ldots, e_m\}$. The inner product between $x, y \in \mathbb{R}^m$ is denoted by $\langle x, y\rangle$. The 2-norm of a point $x \in \mathbb{R}^m$ is $\|x\|_2 = \sqrt{\langle x, x\rangle}$, and the Euclidean metric $d_E : \mathbb{R}^m \times \mathbb{R}^m \to [0, \infty)$ is defined by $d_E(x, y) = \|x - y\|_2$ for all $x, y \in \mathbb{R}^m$.

The following notations, conventions, and definitions will also be used throughout this paper.

(1) A convex body is a compact convex subset of $\mathbb{R}^m$ with non-empty interior.
(2) For a set B in $\mathbb{R}^m$, the notation $conv(B)$ is used to denote the convex hull of B.

(3) For a set $B \in \mathbb{R}^m$, the *affine hull*, denoted $\text{aff}(B)$, of B is the smallest affine subspace containing B, i.e., the intersection of all affine subspaces containing B.

(4) The symbol $\mathbb{H}$ will denote the nonempty compact subsets of $\mathbb{R}^m$, and the symbol $d_{\mathbb{H}}$ will denote the Hausdorff metric on $\mathbb{H}$. Recall that $(\mathbb{R}^m, d_{\mathbb{H}})$ is a complete metric space.

(5) A metric d on $\mathbb{R}^m$ is said to be *Lipschitz equivalent* to d_E if there are positive constants r and R such that

$$r\, d_E(x, y) \leq d(x, y) \leq R\, d_E(x, y),$$

for all $x, y \in \mathbb{R}^m$. If two metrics are Lipschitz equivalent, then they induce the same topology on $\mathbb{R}^m$, but the converse is not necessarily true.

(6) For any two subsets A and B of $\mathbb{R}^m$, the notation $A - B := \{x - y : x \in A \text{ and } y \in B\}$ is used to denote the pointwise subtraction of elements in the two sets.

(7) For a positive integer N, the symbol $\Omega = \{1, 2, \ldots, N\}^\infty$ will denote the set of all infinite sequences of symbols $\{\sigma_k\}_{k=1}^\infty$ belonging to the alphabet $\{1, 2, \ldots, N\}$. The set Ω is endowed with the product topology. An element of $\sigma \in \Omega$ will also be denoted by the concatenation $\sigma = \sigma_1\sigma_2\sigma_3 \ldots$, where σ_k denotes the k^{th} component of σ. Recall that since Ω is endowed with the product topology, it is a compact Hausdorff space.

Definition 2.1 (IFS)**.** If $N > 0$ is an integer and $f_n : \mathbb{R}^m \to \mathbb{R}^m$, $n = 1, 2, \ldots, N$, are continuous mappings, then $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is called an *iterated function system* (IFS). If each $f \in \mathcal{F}$ is an affine map on $\mathbb{R}^m$, then $\mathcal{F}$ is called an *affine IFS*.

Definition 2.2 (Contractive IFS)**.** An IFS $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is *contractive* when each f_n is a contraction. Namely, there is a number $\alpha_n \in [0, 1)$ such that $d_E(f_n(x), f_n(y)) \leq \alpha_n d_E(x, y)$ for all $x, y \in \mathbb{R}^m$, for all n.

Definition 2.3 (Hyperbolic IFS)**.** An IFS $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is called *hyperbolic* if there is a metric on $\mathbb{R}^m$ Lipschitz equivalent to the given metric so that each f_n is a contraction.

Definition 2.4 (Coding Map)**.** A continuous map $\pi : \Omega \to \mathbb{R}^m$ is called a *coding map* for the IFS $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ if, for each

$n = 1, 2, \ldots, N$, the following diagram commutes,

$$\begin{array}{ccc} \Omega & \overset{s_n}{\rightarrow} & \Omega \\ \pi \downarrow & & \downarrow \pi \\ \mathbb{R}^m & \underset{f_n}{\rightarrow} & \mathbb{R}^m \end{array} \tag{2.1}$$

where the symbol $s_n : \Omega \to \Omega$ denotes the inverse shift map defined by $s_n(\sigma) = n\sigma$.

The notion of a coding map is due to Jun Kigami [11] and Kameyama [9].

Definition 2.5 (Point-Fibered IFS)**.** An IFS $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is *point-fibered* if, for each $\sigma = \sigma_1\,\sigma_2\,\sigma_3 \cdots \in \Omega$, the limit on the right hand side of

$$\pi(\sigma) := \lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x) \tag{2.2}$$

exists, is independent of $x \in \mathbb{R}^m$ for fixed σ, and the map $\pi : \Omega \to \mathbb{R}^m$ is a coding map.

It is not difficult to show that (2.2) is the unique coding map of a point-fibered IFS. Our notion of a point-fibered iterated function system is similar to that of Kieninger [10, Definition 4.3.6, p. 97]. However, we work in the setting of complete metric spaces whereas Kieninger frames his definition in a compact Hausdorff space.

Definition 2.6 (The Symbol $\mathcal{F}(B)$ for an IFS)**.** For an IFS $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ define $\mathcal{F} : \mathbb{H} \to \mathbb{H}$ by

$$\mathcal{F}(B) = \bigcup_{n=1}^{N} f_n(B).$$

(The same symbol $\mathcal{F}$ is used for both the IFS and the mapping.) For $B \in \mathbb{H}$, let $\mathcal{F}^{\circ k}(B)$ denote the k-fold composition of $\mathcal{F}$, i.e., the union of $f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(B)$ over all words $\sigma_1\sigma_2 \cdots \sigma_k$ of length k.

Definition 2.7 (Attractor for an IFS)**.** A set $A \in \mathbb{H}$ is called an *attractor* of an IFS $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ if

$$A = \mathcal{F}(A) \text{ and} \tag{2.3}$$

$$A = \lim_{k\to\infty} \mathcal{F}^{\circ k}(B), \tag{2.4}$$

the limit with respect to the Hausdorff metric, for all $B \in \mathbb{H}$.

If an IFS has an attractor A, then clearly A is the unique attractor. It is well known that a hyperbolic IFS has an attractor. An elegant proof of this fact is given by John E. Hutchinson [7]. He observes that a contractive IFS $\mathcal{F}$ induces a contraction $\mathcal{F} : \mathbb{H} \to \mathbb{H}$, from which the result follows by the contraction mapping theorem. See also [6] and [19].

Section 4 shows that a point-fibered IFS $\mathcal{F}$ has an attractor A, and, moreover, if π is the coding map of $\mathcal{F}$, then $A = \pi(\Omega)$. Often σ is considered as the "address" of the point $\pi(\sigma)$ in the attractor. In the literature on fractals (for example, [11]) there is an approach to the concept of a self-similar system without reference to the ambient space. This approach begins with the idea of a continuous coding map π and, in effect, defines the attractor as $\pi(\Omega)$.

3. Examples and remarks on iterated function systems

This section contains examples and remarks relevant to Theorem 1.1 and Theorem 1.2.

Example 3.1 (A Point-Fibered, Not Contractive IFS)**.** Consider the affine IFS consisting of a single linear function on $\mathbb{R}^2$ given by the matrix

$$f = \begin{pmatrix} 0 & 2 \\ \frac{1}{8} & 0 \end{pmatrix}.$$

Note that the eigenvalues of f equal $\pm\frac{1}{2}$. Since

$$\lim_{n\to\infty} f^{\circ 2n} = \lim_{n\to\infty} T^{-1} \begin{pmatrix} (\frac{1}{2})^n & 0 \\ 0 & (-\frac{1}{2})^n \end{pmatrix} T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where T is the change of basis matrix, this IFS is point-fibered. However, since

$$f \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

the mapping is not a contraction under the usual metric on $\mathbb{R}^2$. Theorem 1.1, however, guarantees we can remetrize $\mathbb{R}^2$ with an equivalent metric so that f is a contraction.

Example 3.2 (An IFS with Point-Fibered Functions That Is Not Point-Fibered)**.** In the literature on affine iterated function systems, it is sometimes assumed that the eigenvalues of the linear parts of the affine functions are less than 1 in modulus. Unfortunately, this assumption is not sufficient to imply any of the five statements given in Theorem 1.1. While the affine IFS $(\mathbb{R}^m; f)$ is point-fibered if and only if the eigenvalues of the linear part of f all have moduli strictly less than 1, an analogous statement cannot be made if the number of functions in the IFS is larger than 1.

Consider the affine IFS $\mathcal{F} = \left(\mathbb{R}^2; f_1, f_2\right)$, where

$$f_1 = \begin{pmatrix} 0 & 2 \\ \frac{1}{8} & 0 \end{pmatrix} \qquad \text{and} \qquad f_2 = \begin{pmatrix} 0 & \frac{1}{8} \\ 2 & 0 \end{pmatrix}.$$

As noted in Example 3.1,

$$\lim_{n\to\infty} f_1^{\circ n}\mathbf{u} = \lim_{n\to\infty} f_2^{\circ n}\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for any vector $\mathbf{u}$. Thus, both $\mathcal{F}_1 = \left(\mathbb{R}^2; f_1\right)$ and $\mathcal{F}_2 = \left(\mathbb{R}^2; f_2\right)$ are point-fibered. Unfortunately, their product is the matrix

$$f_1 \circ f_2 = \begin{pmatrix} 4 & 0 \\ 0 & \frac{1}{64} \end{pmatrix},$$

so that

$$\lim_{n\to\infty} (f_1 \circ f_2)^{\circ n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lim_{n\to\infty} \begin{pmatrix} 4^n \\ 0 \end{pmatrix} = +\infty\,.$$

Thus, the IFS $\mathcal{F} = \left(\mathbb{R}^2; f_1, f_2\right)$ fails to be point-fibered.

Remark 3.3. While it is true that (1) $\Rightarrow$ (2) in Theorem 1.1 even without the assumption that the IFS is affine, the converse is not true in general. Kameyama [9] has shown that there exists a point-fibered IFS that is not hyperbolic. We next give an example of an affine IFS with a coding map that is not point-fibered. Thus, the set of IFSs (with a coding map) strictly contains the set of point-fibered IFSs which, in turn, strictly contains the set of hyperbolic IFSs.

Example 3.4 (The Failure of a Finite Eigenvalue Test to Imply Point-Fibered)**.** Consider the linear IFS $\mathcal{F} = \left(\mathbb{R}^2; L_1, L_2\right)$, where

$$L_1 = \begin{pmatrix} 0 & 2 \\ \frac{1}{8} & 0 \end{pmatrix} \qquad \text{and} \qquad L_2 = \begin{pmatrix} a\cos\theta & -a\sin\theta \\ a\sin\theta & a\cos\theta \end{pmatrix} = aR_\theta,$$

where R_θ denotes rotation by angle θ, and $0 < a < 1$. Then L_1^n has eigenvalues $\pm 1/2^n$ while the eigenvalues of L_2^n both have magnitude

$a^n < 1$. For example, if we choose $\theta = \pi/8$ and $a = 31/32$, then it is readily verified that the eigenvalues of L_1L_2 and L_2L_1 are smaller than one in magnitude and that one of the eigenvalues of $L_1L_2L_2$ is 1.4014... . Hence, in this case, the magnitudes of the eigenvalues of the linear operators L_1, L_2, L_1^2, L_1L_2, L_2L_1, L_2^2 are all less than one, but $\|(L_1L_2L_2)^n\, x\|$ does not converge when $x \in \mathbb{R}^2$ is any eigenvector of $L_1L_2L_2$ corresponding to the eigenvalue 1.4014... . It follows that the IFS $\left(\mathbb{R}^2; L_1, L_2\right)$ is not point-fibered. By using the same underlying idea, it is straightforward to prove that, given any positive integer M, we can choose a close to 1 and θ close to 0 in such a way that the eigenvalues of $L_{\sigma_1}L_{\sigma_2}...L_{\sigma_k}$, (where $\sigma_j \in \{1,2\}$ for $j = 1, 2, ..., k$, with $k \leq M$) are all of magnitude less than one, while $L_1L_2^M$ has an eigenvalue of magnitude larger than one.

Example 3.5 (A Non-Hyperbolic Affine IFS)**.** Let $\mathcal{F} = \left(\mathbb{R}^2; f_0, f_1\right)$, where

$$f_0(x_1, x_2) = (\frac{1}{2}\, x_1, x_2), \qquad\qquad f_1(x_1,\, x_2) = (\frac{1}{2}\, x_1 + \frac{1}{2},\, x_2).$$

This IFS has a coding map π with $\Omega = \{0,1\}^\infty$ and $\pi(\sigma) = (0.\sigma, 0)$, where $0.\sigma$ is considered as a base 2 decimal. Since $\lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x_1, x_2) = (0.\sigma, x_2)$ depends on the choice of the points $(x_1, x_2) \in \mathbb{R}^2$, this IFS cannot be point-fibered. Hence, by Theorem 1.1, the IFS $\mathcal{F}$ is also not hyperbolic. However, it is clearly hyperbolic when restricted to the x-axis, the affine hull of unit interval $\pi(\Omega) = [0,1] \times \{0\}$. Thus, this example illustrates Theorem 1.2.

Remark 3.6. A key fact used in the proof of Theorem 1.1 is that the set of antipodal points in a convex body equals the set of diametric points. The definitions of antipodal and diametric points are given in definitions 6.1 and 6.2, respectively. The equality between these two point sets is proved in Theorem 6.4. While it is possible that this result is present in the convex geometry literature, it does not seem to be well known. For example, it is not mentioned in the works of Maria Moszyńska [13] or Rolf Schneider [17]. This equivalence between antipodal and diametric points is crucial to our work because it provides the remetrization technique at the heart of Theorem 6.7, which implies that a non-antipodal IFS is hyperbolic. A consequence of Theorem 1.1 is that a non-antipodal affine IFS has the seemingly stronger property of being topologically contractive.

4. Hyperbolic implies point-fibered implies the existence of an attractor

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) in Theorem 1.1 are proved in this section. We also introduce the notation $f_{\sigma\,|\,k} = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x)$. Note that, for k fixed, $f_{\sigma\,|\,k}(x)$ is a function of both x and σ.

Theorem 4.1. *If $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is a hyperbolic IFS, then $\mathcal{F}$ is point fibered.*

Proof: For $\sigma \in \Omega$, the proof that the limit $\lim_{k\to\infty} f_{\sigma|k}$ exists and is independent of x is virtually identical to the proof of the classical Contraction Mapping Theorem. Moreover, the same proof shows that the limit is uniform in σ.

With $\pi : \Omega \to \mathbb{R}^m$ defined by $\pi(\sigma) = \lim_{k\to\infty} f_{\sigma|k}$, it is easy to check that, for each $n = 1, 2, \ldots, N$, the diagram (2.1) commutes.

It only remains to show that π is continuous. With x fixed, $f_{\sigma\,|\,k}(x)$ is a continuous function of σ. This is simply because, if $\sigma, \tau \in \Omega$ are sufficiently close in the product topology, then they agree on the first k components. By Definition 2.5, the function π is then the uniform limit of continuous (in σ) functions defined on the compact set Ω. Therefore, π is continuous. $\square$

Let $\mathcal{F}$ be a point-fibered affine IFS, and let A denote the set

$$A := \pi(\Omega).$$

According to Theorem 4.3, A is the attractor of $\mathcal{F}$.

Lemma 4.2. *Let $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ be a point-fibered affine IFS with coding map $\pi : \Omega \to \mathbb{R}^m$. If $B \subset \mathbb{R}^m$ is compact, then the convergence in the limit*

$$\pi(\sigma) = \lim_{k\to\infty} f_{\sigma|k}(x)$$

is uniform in $\sigma = \sigma_1\sigma_2 \cdots \in \Omega$ and $x \in B$ simultaneously.

Proof: Only the uniformity requires proof. Express $f_n(x) = L_n x + a_n$, where L_n is the linear part. Then

(4.1)
$$\begin{aligned} f_{\sigma|k}(x) &= L_{\sigma|k}(x) + L_{\sigma|k-1}(a_{\sigma_k}) + L_{\sigma|k-2}(a_{\sigma_{k-1}}) + \cdots + L_{\sigma|1}a_{\sigma_2} + a_{\sigma_1} \\ &= L_{\sigma|k}(x) + f_{\sigma|k}(0). \end{aligned}$$

From equation (4.1) it follows that, for any $x, y \in B$,

$$d_E(f_{\sigma|k}(x), f_{\sigma|k}(y)) = \left\|L_{\sigma|k}(x-y)\right\|_2 \tag{4.2}$$
$$\leq \sup\left\{\sum_{j=1}^{m} 2\left|c_j\right| \left\|L_{\sigma|k}(e_j)\right\|_2 \,:\, c_1e_1 + \cdots + c_me_m \in B\right\}$$
$$\leq c \max_j \left\|f_{\sigma|k}(e_j) - f_{\sigma|k}(0)\right\|_2,$$

where $c = 2m \cdot \sup\{\max_j |c_j| \,:\, c_1e_1 + \cdots + c_me_m \in B\}$ and where $\{e_j\}_{j=1}^m$ is a basis for $\mathbb{R}^m$.

Let $\epsilon > 0$. From the definition of point-fibered there is a k_j, independent of σ, such that if $k > k_j$, then

$$\left\|f_{\sigma|k}(e_j) - \pi(\sigma)\right\|_2 < \frac{\epsilon}{4c} \qquad \text{and} \qquad \left\|f_{\sigma|k}(0) - \pi(\sigma)\right\|_2 < \frac{\epsilon}{4c},$$

which implies $\left\|f_{\sigma|k}(e_j) - f_{\sigma|k}(0)\right\|_2 < \frac{\epsilon}{2c}$. This and equation (4.2) imply that if $k \geq \overline{k} := \max_j k_j$, then, for any $x, y \in B$, we have

$$d_E(f_{\sigma|k}(x), f_{\sigma|k}(y)) < c\frac{\epsilon}{2c} = \frac{\epsilon}{2}\,. \tag{4.3}$$

Let b be a fixed element of B. There is a k_b, independent of σ, such that if $k > k_b$, then $d_E(f_{\sigma|k}(b), \pi(\sigma)) < \frac{\epsilon}{2}$. If $k > max(k_b, \overline{k})$, then, by equation (4.3), for any $x \in B$,

$$d_E(f_{\sigma|k}(x), \pi(\sigma)) \leq d_E(f_{\sigma|k}(x), f_{\sigma|k}(b)) + d_E(f_{\sigma|k}(b), \pi(\sigma)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

Theorem 4.3 (A Point-Fibered IFS Has an Attractor)**.** *If* $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ *is a point-fibered affine IFS, then* $\mathcal{F}$ *has an attractor* $A = \pi(\Omega)$*, where* $\pi : \Omega \to \mathbb{R}^m$ *is the coding map of* $\mathcal{F}$*.*

Proof: It follows directly from the commutative diagram (2.1) that A obeys the self-referential equation (2.3). We next show that A satisfies equation (2.4).

Let $\epsilon > 0$. We must show that there is an M such that if $k > M$, then $d_{\mathbb{H}}(\mathcal{F}^{\circ k}(B), \pi(\Omega)) < \epsilon$. It is sufficient to let $M = \max(M_1, M_2)$, where M_1 and M_2 are defined as follows.

First, let a be an arbitrary element of A. Then there exists a $\sigma \in \Omega$ such that $a = \pi(\sigma)$. By Lemma 4.2, there is an M_1 such that if $k > M_1$, then $d_E(f_{\sigma|k}(b), a) = d_E(f_{\sigma|k}(b), \pi(\sigma)) < \epsilon$, for all $b \in B$. In other words, A lies in an ϵ-neighborhood of $\mathcal{F}^{\circ k}(B)$.

Second, let b be an arbitrary element of B and σ an arbitrary element of Ω. If $a := \pi(\sigma) \in A$, then there is an M_2 such that if $k > M_2$, then $d_E(f_{\sigma|k}(b), a) = d_E(f_{\sigma|k}(b), \pi(\sigma)) < \epsilon$. In other words, $\mathcal{F}^{\circ k}(B)$ lies in an ϵ-neighborhood of A. □

5. An IFS with an attractor is topologically contractive

The goal of this section is to establish the implication (3) ⇒ (4) in Theorem 1.1. We will show that if an affine IFS has an attractor as defined in Definition 2.7, then it is a topological contraction. The proof uses notions involving convex bodies.

Definition 5.1. A convex body K is *centrally symmetric* if it has the property that whenever $x \in K$, then $-x \in K$.

A well-known general technique for creating centrally symmetric convex bodies from a given convex body is provided by the next proposition.

Proposition 5.2. *If a set K is a convex body in $\mathbb{R}^m$, then the set $K' = K - K$ is a centrally symmetric convex body in $\mathbb{R}^m$.*

Definition 5.3 (Minkowski Norm)**.** If K is a centrally symmetric convex body in $\mathbb{R}^m$, then the *Minkowski norm* on $\mathbb{R}^m$ is defined by

$$\|x\|_K = \inf\{\lambda \geq 0 \,:\, x \in \lambda K\}.$$

The next proposition is also well known.

Proposition 5.4. *If K is a centrally symmetric convex body in $\mathbb{R}^m$, then the function $\|x\|_K$ defines a norm on $\mathbb{R}^m$. Moreover, the set K is the unit ball with respect to the Minkowski norm $\|x\|_K$.*

Definition 5.5 (Minkowski Metric)**.** If K is a centrally symmetric convex body in $\mathbb{R}^m$ and $\|x\|_K$ is the associated Minkowski norm, then define the *Minkowski metric* on $\mathbb{R}^m$ by the rule

$$d_K(x, y) := \|x - y\|_K\,.$$

While R. Tyrrel Rockafellar [15] refers to such a metric as a *Minkowski metric*, the reader should be aware that this term is also associated with the metric on space-time in the theory of relativity. Since, for any convex body K, there are positive numbers r and R such that K contains a ball of radius r and is contained in a ball of radius R, the following proposition is clear.

Proposition 5.6. *If d is a Minkowski metric, then d is Lipschitz equivalent to the standard metric d_E on $\mathbb{R}^m$.*

Proposition 5.7. *A metric $d : \mathbb{R}^m \times \mathbb{R}^m \to [0, \infty)$ is a Minkowski metric if and only if it is translation invariant and distances behave linearly along line segments. More specifically,*

$$d(x+z, y+z) = d(x,y) \text{ and } d(x, (1-\lambda)x + \lambda y) = \lambda d(x,y) \tag{5.1}$$

for all $x, y, z \in \mathbb{R}^m$ and all $\lambda \in [0,1]$.

Proof: For a proof, see [15, pp. 131–132]. □

Definition 5.8 (Topologically Contractive IFS)**.** An IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, f_2, ..., f_N\}$ is called *topologically contractive* if there is a convex body K such that $\mathcal{F}(K) \subset int(K)$.

The proof of Theorem 5.10 relies on the following lemma which is easily proved.

Lemma 5.9. *If $g : \mathbb{R}^m \to \mathbb{R}^m$ is affine and $S \subset \mathbb{R}^m$, then $g(conv(S)) = conv(g(S))$.*

Theorem 5.10 (The Existence of an Attractor Implies a Topological Contraction)**.** *For an affine IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, f_2, ..., f_N\}$, if there exists an attractor $A \in \mathbb{H}$ of the affine IFS $\mathcal{F} = \{\mathbb{R}^m; f_1, f_2, ..., f_N\}$, then $\mathcal{F}$ is topologically contractive.*

Proof: The proof of this theorem unfolds in three steps.

(1) There exists a convex body K_1 and a positive integer t with the property that $\mathcal{F}^{\circ t}(K_1) \subset int(K_1)$.

(2) The set K_1 is used to define a convex body K_2 such that $L_n(K_2) \subset int(K_2)$, where $f_n(x) = L_n x + a_n$ and $n = 1, 2, \ldots, N$.

(3) There is a positive constant c such that the set $K = cK_2$ has the property $\mathcal{F}(K) \subset int(K)$.

(1): Let A denote the attractor of $\mathcal{F}$. Let $A_\rho = \{x \in \mathbb{R}^m : d_{\mathbb{H}}(\{x\}, A) \leq \rho\}$ denote the dilation of A by radius $\rho > 0$. Since we are assuming $\lim_{k\to\infty} d_{\mathbb{H}}(\mathcal{F}^{\circ k}(A_\rho), A) = 0$, we can find an integer t so that $d_{\mathbb{H}}(\mathcal{F}^{\circ t}(A_1), A) < 1$. Thus,

$$\mathcal{F}^{\circ t}(A_1) \subseteq int(A_1). \tag{5.2}$$

If we let $K_1 := conv\,(A_1)$, then

$$\begin{aligned}
\mathcal{F}^{\circ t}\,(K_1) &= \bigcup_{i_1\in\Omega}\ \bigcup_{i_2\in\Omega}\cdots\bigcup_{i_t\in\Omega}(f_{i_1}\circ f_{i_2}\circ\cdots\circ f_{i_t})\,(conv(A_1))\\
&= \bigcup_{i_1\in\Omega}\ \bigcup_{i_2\in\Omega}\cdots\bigcup_{i_t\in\Omega} conv\,(f_{i_1}\circ f_{i_2}\circ\cdots\circ f_{i_t}(A_1))\\
&\qquad \text{(by Lemma 5.9)}\\
&\subseteq \bigcup_{i_1\in\Omega}\ \bigcup_{i_2\in\Omega}\cdots\bigcup_{i_t\in\Omega} conv\,(int(A_1))\\
&= conv(int(A_1))\\
&\qquad \text{(by inclusion (5.2))}\\
&\subseteq int(conv(A_1)) = int\,(K_1)\,.
\end{aligned}$$

This argument completes the proof of (1).

(2): Consider the set
$$K_2 := \textstyle\sum_{k=0}^{t-1}(conv(\mathcal{F}^{\circ k}(K_1) - conv(\mathcal{F}^{\circ k}(K_1))).$$
The set K_2 is a centrally symmetric convex body because it is a finite Minkowski sum of centrally symmetric convex bodies. If any affine map f_n in $\mathcal{F}$ is written $f_n(x) = L_n x + a_n$, where $L_n : \mathbb{R}^m \to \mathbb{R}^m$ denotes the linear part, then

$$\begin{aligned}
L_n(K_2) &= \sum_{k=0}^{t-1} L_n\Big(conv(\mathcal{F}^{\circ k}(K_1) - conv(\mathcal{F}^{\circ k}(K_1))\Big)\\
&\qquad \text{(since } L_n \text{ is a linear map)}\\
&= \sum_{k=0}^{t-1}\Big(conv(L_n\left(\mathcal{F}^{\circ k}\,(K_1)\right)) - conv(L_n\left(\mathcal{F}^{\circ k}(K_1)\right))\Big)\\
&\qquad \text{(by Lemma 5.9)}\\
&= \sum_{k=0}^{t-1}\Big(conv(f_n\left(\mathcal{F}^{\circ k}\,(K_1)\right)) - conv(f_n\left(\mathcal{F}^{\circ k}(K_1)\right))\Big)\\
&\qquad \text{(since the } a_n\text{s cancel)}\\
&\subseteq \sum_{k=0}^{t-1}\Big(conv(\mathcal{F}^{\circ (k+1)}\,(K_1)) - conv(\mathcal{F}^{\circ (k+1)}(K_1))\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \Big(conv(\mathcal{F}^{\circ t}(K_1) - conv(\mathcal{F}^{\circ t}(K_1))\Big) \\
&\qquad + \sum_{k=1}^{t-1}\Big(conv(\mathcal{F}^{\circ k}(K_1) - conv(\mathcal{F}^{\circ k}(K_1))\Big) \\
&\subseteq (int\,(K_1) - int\,(K_1)) \\
&\qquad + \sum_{k=1}^{t-1}(conv(\mathcal{F}^{\circ k}(K_1) - conv(\mathcal{F}^{\circ k}(K_1))) \\
&\qquad \text{(by step (1))} \\
&= int(K_2).
\end{aligned}
$$

The second to last inclusion follows from the fact that $f_n\left(\mathcal{F}^{\circ k}(K_1)\right) \subset \mathcal{F}^{\circ(k+1)}(K_1)$. The last equality follows from the fact that if $\mathcal{O}$ and $\mathcal{C}$ are symmetric convex bodies in $\mathbb{R}^m$, then $int(\mathcal{O}) + \mathcal{C} = int\,(\mathcal{O} + \mathcal{C})$. We have now completed the proof of step (2).

(3): It follows from step (2) and the compactness of K_2 that there is a constant $\alpha \in (0,1)$ such that $d_{K_2}(L_n(x), L_n(y)) < \alpha d_{K_2}(x,y)$ for all $x, y \in \mathbb{R}^m$ and all $n = 1, 2, \ldots, N$.

Let

$$c > \frac{r}{(1-\alpha)}\ ,$$

where $r = \max\{d_{K_2}(a_1, 0), d_{K_2}(a_2, 0), \ldots, d_{K_2}(a_N, 0)\}$. If $x \in cK_2$ and $f(x) = Lx + a$ is any function in the IFS $\mathcal{F}$, then

$$
\begin{aligned}
\|f(x)\|_{K_2} &= d_{K_2}(f(x), 0) \\
&= d_{K_2}(Lx + a, 0) \leq d_{K_2}(Lx + a, Lx) + d_{K_2}(Lx, 0) \\
&= d_{K_2}(a, 0) + d_{K_2}(Lx, 0) \quad \text{(by equation (5.1))} \\
&< r + \alpha d_{K_2}(x, 0) = r + \alpha \|x\|_{K_2} \\
&\leq r + \alpha c < (c - \alpha c) + \alpha c = c.
\end{aligned}
$$

This inequality shows that $\mathcal{F}(cK_2) \subset int(cK_2)$. $\square$

6. A non-antipodal affine IFS is hyperbolic

Let $S^{m-1} \subset \mathbb{R}^m$ denote the unit sphere in $\mathbb{R}^m$. For a convex body $K \subset \mathbb{R}^m$ and for $u \in S^{m-1}$ there exists a pair $\{H_u, H_{-u}\}$ of distinct supporting hyperplanes of K, each orthogonal to u and with the property that they both intersect ∂K but contain no points of the interior of K. The pair $\{H_u, H_{-u}\}$ is usually referred to as the two *supporting hyperplanes* of K orthogonal to u. (See [13, p. 14].)

Definition 6.1 (Antipodal Pairs)**.** If $K \subset \mathbb{R}^m$ is a convex body and $u \in S^{m-1}$, then define

$$\begin{aligned} \mathcal{A}_u &:= \mathcal{A}_u(K) = (H_u \cap \partial K) \times (H_{-u} \cap \partial K) \quad \text{and} \\ \mathcal{A} &:= \mathcal{A}(K) = \bigcup_{u \in S^{m-1}} \mathcal{A}_u. \end{aligned}$$

We say that (p, q) is an *antipodal pair* of points with respect to K if $(p, q) \in \mathcal{A}$.

Definition 6.2 (Diametric Pairs)**.** If $K \subset \mathbb{R}^m$ is a convex body, and $u \in S^{m-1}$, then define the *diameter* of K *in the direction* u to be

$$D(u) = \max\{\|x - y\|_2 : x, y \in K, x - y = \alpha u, \alpha \in \mathbb{R}\}.$$

The maximum is achieved at some pair of points belonging to ∂K because $K \times K$ is convex and compact, and $\|x - y\|_2$ is continuous for $(x, y) \in K \times K$. Now define

$$\begin{aligned} \mathcal{D}_u &= \{(p, q) \in \partial K \times \partial K : D(u) = \|q - p\|_2\} \quad \text{and} \\ \mathcal{D} &= \bigcup_{u \in S^{m-1}} \mathcal{D}_u. \end{aligned}$$

We say that $(p, q) \in \mathcal{D}_u$ is a *diametric pair* of points in the direction of u and that $\mathcal{D}$ is the set of diametric pairs of points of K.

Definition 6.3 (Strictly Convex)**.** A convex body K is *strictly convex* if, for every two points $x, y \in K$, the open line segment joining x and y is contained in the interior of K.

We write xy to denote the closed line segment with endpoints at x and y so that $y - x$ is the vector, in the direction from x to y, whose magnitude is the length of xy.

Theorem 6.4. *If $K \subset \mathbb{R}^m$ is a convex body, then the set of antipodal pairs of points of K is the same as the set of diametric pairs of points of K, i.e.,*

$$\mathcal{A} = \mathcal{D}.$$

Proof: First, we show that $\mathcal{A} \subseteq \mathcal{D}$. If $(p, q) \in \mathcal{A}$, then $p \in H_u \cap \partial K$ and $q \in H_{-u} \cap \partial K$ for some $u \in S^{m-1}$. Clearly, any chord of K parallel to pq lies entirely in the region between H_u and

H_{-u} and therefore cannot have length greater than that of pq. So $D(q-p) = \|q-p\|$ and $(p,q) \in \mathcal{D}_{q-p} \subseteq \mathcal{D}$. For use later in the proof, note that if K is strictly convex, then pq is the unique chord of maximum length in its direction.

Conversely, to show that $\mathcal{D} \subseteq \mathcal{A}$, first consider the case where K is a strictly convex body. For each $u \in S^{m-1}$, consider the points $x_u \in H_u \cap \partial K$ and $x_{-u} \in H_{-u} \cap \partial K$. The continuous function $f : S^{m-1} \to S^{m-1}$ defined by

$$f(u) = \frac{x_u - x_{-u}}{\|x_u - x_{-u}\|_2}$$

has the property that $\langle f(u), u\rangle > 0$ for all u. In other words, the angle between u and $f(u)$ is less than $\frac{\pi}{2}$. But it is an elementary exercise in topology (see, for example, [14, Problem 10, p. 367]) that if $f : S^{m-1} \to S^{m-1}$ maps no point x to its antipode $-x$, then f has degree 1 and, in particular, is surjective. To show that $\mathcal{D} \subseteq \mathcal{A}$, let $(p,q) \in \mathcal{D}_v$ for some $v \in S^{m-1}$. By the surjectivity of f there is $u \in S^{m-1}$ such that $f(u) = v$. According to the last sentence of the previous paragraph, $x_u x_{-u}$ is the unique longest chord parallel to v. Therefore, $p = x_u$ and $q = x_{-u}$ and consequently, $(p,q) \in \mathcal{A}_u$.

The case where K is not strictly convex is treated by a standard limiting argument. Given a vector $v \in S^{m-1}$ and a longest chord pq parallel to v, we must prove that $(p,q) \in \mathcal{A}$. Since K is the intersection of all strictly convex bodies containing K, there is a sequence $\{K_k\}$ of strictly convex bodies containing K with the following two properties.

(1) There is a longest chord $p_k q_k$ of K_k parallel to u such that $\lim_{k\to\infty} \|p_k - q_k\|_2 = \|p-q\|_2$, and the limits $\lim_{k\to\infty} p_k = \tilde{p} \in K$ and $\lim_{k\to\infty} q_k = \tilde{q} \in K$ exist.

By the result for the strictly convex case, there is a sequence of vectors $u_k \in S^{m-1}$ such that $p_k = K_k \cap H_{u_k}(K_k)$ and $q_k = K_k \cap H_{-u_k}(K_k)$. By perhaps going to a subsequence

(2) $\lim_{k\to\infty} u_k = u \in S^{m-1}$ exists.

It follows from item (1) that $\|\tilde{p} - \tilde{q}\|_2 = \|p-q\|_2$ and $\tilde{p} - \tilde{q}$ is parallel to v. Therefore, $\tilde{p}\tilde{q}$, as well as pq, are longest chords of K parallel to v. It follows from (2) that if H and H' are the hyperplanes orthogonal to u through $\tilde{p}$ and $\tilde{q}$, respectively, then H and H' are parallel supporting hyperplanes of K. Therefore, necessarily $p \in H$ and $q \in H'$, and consequently, $(p, q) \in \mathcal{A}_u \subset \mathcal{A}$. $\square$

Definition 6.5 (Non-Antipodal IFS)**.** If $K \subset \mathbb{R}^m$ is a convex body, then $f : \mathbb{R}^m \to \mathbb{R}^m$ is *non-antipodal* with respect to K if $f(K) \subseteq K$, and $(x, y) \in \mathcal{A}(K)$ implies $(f(x), f(y)) \notin \mathcal{A}(K)$. If $\mathcal{F} = \{\mathbb{R}^m; f_1, f_2, ..., f_N\}$ is an iterated function system with the property that each f_n is non-antipodal with respect to K, then $\mathcal{F}$ is called *non-antipodal* with respect to K.

The next proposition gives the implication (4) $\Rightarrow$ (5) in Theorem 1.1. The proof is clear.

Proposition 6.6 (A Topological Contraction Is Non-Antipodal)**.** *If $\mathcal{F} = \{\mathbb{R}^m; f_1, f_2, ..., f_N\}$ is an affine iterated function system with the property that there exists a convex body $K \subset \mathbb{R}^m$ such that $f_n(K) \subset int(K)$ for all $n = 1, 2, \ldots, n$, then $\mathcal{F}$ is non-antipodal with respect to K.*

The next theorem provides the implication (5) $\Rightarrow$ (1) in Theorem 1.1.

Theorem 6.7. *If the affine IFS $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is non-antipodal with respect to a convex body K, then $\mathcal{F}$ is hyperbolic.*

Proof: Assume that K is a convex body such that f is non-antipodal with respect to K for all $f \in \mathcal{F}$. Let $C = K - K$ and let $f(x) = Lx + a \in \mathcal{F}$, where L is the linear part of f. By Proposition 5.2, the set C is a centrally symmetric convex body and

$$L(C) = L(K) - L(K) = f(K) - f(K) \subseteq K - K = C.$$

We claim that $L(C) \subset int(C)$. Since C is compact and L is linear, to prove the claim, it is sufficient to show that $L(x) \notin \partial C$ for all $x \in \partial C$. By way of contradiction, assume that $x \in \partial C$ and $L(x) \in \partial C$. Then the vector x is a longest vector in C in its direction. Since $x \in C = K - K$, there are $x_1, x_2 \in \partial K$ such that $x = x_1 - x_2$, and $(x_1, x_2) \in \mathcal{D}(K) = \mathcal{A}(K)$, where the last equality is by Theorem 6.4. So (x_1, x_2) is an antipodal pair with respect to K. Likewise, since Lx is a longest vector in C in its direction, there are $y_1, y_2 \in \partial K$ such that $Lx = y_1 - y_2$, and $(y_1, y_2) \in \mathcal{D}(K) = \mathcal{A}(K)$.

Therefore,

$$f(x_2) - f(x_1) = L(x_2) - L(x_1) = L(x_2 - x_1) = Lx = y_1 - y_2,$$

which implies that $(f_n(x_1), f_n(x_2)) \in \mathcal{D}(K) = \mathcal{A}(K)$, contradicting that f is non-antipodal with respect to K.

If d_C denotes the Minkowski metric with respect to the centrally symmetric convex body C, then by Proposition 5.4, C is the unit ball centered at the origin with respect to this metric. Since C is compact, the containment $L(C) \subset int(C)$ implies that there is an $\alpha \in [0, 1)$ such that $\|Lx\|_C < \alpha \, \|x\|_C$ for all $x \in \mathbb{R}^m$. Then

$$\begin{aligned} d_C(f(x), f(y)) &= \|f(x) - f(y)\|_C = \|Lx - Ly\|_C \\ &= \|L(x - y)\|_C < \alpha \, \|x - y\|_C = \alpha \, d_C(x, y). \end{aligned}$$

Therefore, d_C is a metric for which each function in the IFS is a contraction. By Proposition 5.6, d_C is Lipschitz equivalent to the standard metric. □

7. An answer to the question of Kameyama

We now turn to the proof of Theorem 1.2, the theorem that settles the question of Kameyama. If $X \subseteq \mathbb{R}^m$ and $\mathcal{F} = (X; f_1, f_2, ..., f_N)$ is an IFS on X, then the definitions of *coding map* and *point-fibered* for $\mathcal{F}$ are exactly the same as definitions 2.4 and 2.5, with $\mathbb{R}^m$ replaced by X. The proof of Theorem 1.2 requires the following proposition.

Proposition 7.1. *If $X \subseteq \mathbb{R}^m$ and $\mathcal{F} = (X; f_1, f_2, ..., f_N)$ is an IFS with a coding map $\pi : \Omega \to \mathbb{R}^m$ such that $\pi(\Omega) = X$, then $\mathcal{F}$ is point-fibered on X.*

Proof: By Definition 2.5, we must show that $\lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x)$ exists, is independent of $x \in X$, and is continuous as a function of $\sigma = \sigma_1\sigma_2 \cdots \in \Omega$. We will actually show that $\lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x) = \pi(\sigma)$.

Since π is a coding map, we know by Definition 2.4 that $f_n \circ \pi(\sigma) = \pi \circ s_n(\sigma)$, for all $n = 1, 2, \ldots, N$. By assumption, if x is any point in X, then there is a $\tau \in \Omega$ such that $\pi(\tau) = x$. Thus,

$$\lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x) = \lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(\pi(\tau)) \quad \text{(since } \pi(\tau) = x\text{)}$$

$$= \lim_{k\to\infty} \pi(s_{\sigma_1} \circ s_{\sigma_2} \circ ... \circ s_{\sigma_k} \circ \tau)$$
(by diagram (2.1))
$$= \pi(\lim_{k\to\infty} s_{\sigma_1} \circ s_{\sigma_2} \circ ... \circ s_{\sigma_k} \circ \tau)$$
(since π is continuous)
$$= \pi(\sigma). \qquad \square$$

Theorem 7.2. *If $\mathcal{F} = (\mathbb{R}^m; f_1, f_2, ..., f_N)$ is an affine IFS with a coding map $\pi : \Omega \to X$, then $\mathcal{F}$ is point-fibered when restricted to the affine hull of $\pi(\Omega)$. In particular, if $\pi(\Omega)$ contains a non-empty open subset of $\mathbb{R}^m$, then $\mathcal{F}$ is point-fibered on $\mathbb{R}^m$.*

Proof: Let $A := \pi(\Omega)$. Since $f_n(A) \subseteq A$ for all n, the restriction of the IFS $\mathcal{F}$ to A, namely $\mathcal{F}|_A := (A; f_1, f_2, \ldots, f_N)$, is well defined. It follows from Proposition 7.1 that $\mathcal{F}|_A$ is point-fibered and, because the coding map for a point-fibered IFS is unique,

$$\pi(\sigma) = \lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(a)$$

for $(\sigma, a) \in \Omega \times A$. It only remains to show that the restriction $\mathcal{F}|_{\text{aff}(A)} := (\text{aff}(A); f_1, f_2, \ldots, f_N)$ of the affine IFS $\mathcal{F}$ to the affine hull of A is point-fibered.

Let $x \in \text{aff}(A)$, the affine hull of A. It is well known that any point in the affine hull can be expressed as a sum, $x = \sum_{p=0}^{m} \lambda_p a_p$ for some $\lambda_0, \lambda_1, ..., \lambda_m \in \mathbb{R}$ such that $\sum_{p=0}^{m} \lambda_p = 1$ and $a_0, a_1, ..., a_m \in A$. Hence, for $(\sigma, x) \in \Omega \times \text{aff}(A)$,

$$\begin{aligned}
\lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(x) &= \lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(\sum_{p=0}^{m} \lambda_p a_p), \\
&= \lim_{k\to\infty} \sum_{p=0}^{m} \lambda_p f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(a_p) \\
&= \sum_{p=0}^{m} \lambda_p \pi(\sigma) = \pi(\sigma). \qquad \square
\end{aligned}$$

Theorem 1.2 now follows easily from Theorem 7.2 and Theorem 1.1.

Proof of Theorem 1.2: Let $A := \pi(\Omega)$ and let $\dim \text{aff}(A) = k \leq m$. It is easy to check from the commuting diagram (2.1) that $f(A) \subseteq A$ for each $f \in \mathcal{F}$ implies that $f(\text{aff}(A)) \subseteq \text{aff}(A)$ for each $f \in \mathcal{F}$. Since aff(A) is isomorphic to $\mathbb{R}^k$, Theorem 1.1 can be applied to the IFS $\mathcal{F}|_{\text{aff}(A)} := (\text{aff}(A); f_1, f_2, ...f_N)$ to conclude that, since it is point-fibered, $\mathcal{F}|_{\text{aff}(A)}$ is also hyperbolic. $\square$

Note that the IFS $(\mathbb{R}; f)$, where $f(x) = 2x + 1$, is not hyperbolic on $\mathbb{R}$, but it is hyperbolic on the affine subspace $\{-1\} \subset \mathbb{R}$.

8. Concluding remarks

Recently it has come to our attention that another condition, equivalent to conditions $(1) - (5)$ in Theorem 1.1, is (6) $\mathcal{F}$ *has joint spectral radius less than one*. (We define the joint spectral radius (JSR) of an affine IFS to be the joint spectral radius of the set of linear factors of its maps.) This information is important because it connects our approach to the rapidly growing literature about JSR; see, for example, [3], [5], and works that refer to these.

Since Example 3.3 and the results presented by Vincent Blondel, Jacques Theys, and Alexander Vladimirov [4] indicate there is no general fast algorithm which will determine whether or not the joint spectral radius of an IFS is less than one, we believe that Theorem 1.1 is important because it provides an easily testable condition that an IFS has a unique attractor. In particular, the topologically contractive and non-antipodal conditions (conditions (4) and (5)) provide geometric/visual tests, which can easily be checked for any affine IFS. In addition to yielding the existence of an attractor, these two conditions also provide information concerning the location of the attractor. (For example, the attractor is a subset of a particular convex body.) We also anticipate that Theorem 1.1 can be generalized into other broader classes of functions, where the techniques developed for the theory of joint spectral radius will not apply.

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