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Indexing the aperture 3 hexagonal discrete global grid

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Abstract

Over the past decade there has been interest in the computer representation of global data based on multi-resolution subdivisions of regular polyhedra. A simple and efficient indexing of the cells of such a subdivision, called A3-coordinates, is introduced. These can be used to encode the $4 \cdot 3^n + 2$ cells at the nth level of resolution of the octahedral aperture 3 hexagonal discrete global grid using $n + 3$ digits, each digit from the set $\{-1, 0, 1\}$.

Keywords: Discrete global grid; Spherical tessellation; Hexagonal tessellation

1. Introduction

While research on mapping the earth goes back at least 2000 years, over the past decade there has been an interest in the computer representation of global data based on multi-resolution subdivisions of regular polyhedra [1–8,10–12,14]. One of the most utilized is referred to as the *aperture 3 hexagonal discrete global grid* (A3HDGG). This is a hierarchical sequence of progressively finer tessellations of a regular polyhedron or sphere into mainly hexagons. Aperture 3 refers to the approximate ratio between the areas of hexagons in successive tessellations in the sequence. In fact, this small ratio is one of the features that makes A3HDGG appealing. There are various ways that the A3HDGG can be realized on the surface of the sphere, the goal being to minimize distortion. One of the most frequently used is the Snyder projection, which is referred to as ISEA3H—*icosahedral Snyder equal area aperture 3 hexagonal discrete global grid* [13]. Fig. 1 shows a few levels of resolution.

The purpose of this paper is to suggest a computer efficient method for indexing the cells in the A3HDGG. Although the method applies to subdivisions of the tetrahedron, octahedron, or icosahedron, it is especially simple for the octahedron. So this paper will mainly focus on the octahedral aperture 3 hexagonal discrete global grid, hereafter referred to as OA3HDGG. Our method is based on an elementary investigation of the properties of the Cartesian coordinates of the barycenters of the cells in OA3HDGG.

The tessellation at the nth level of resolution in the OA3GDGG hierarchy has $4 \cdot 3^n + 2$ cells. Our indexing scheme references each cell at this level by first, what we call its A3-coordinates. This is an ordered triple (a, b, c) of integers such that

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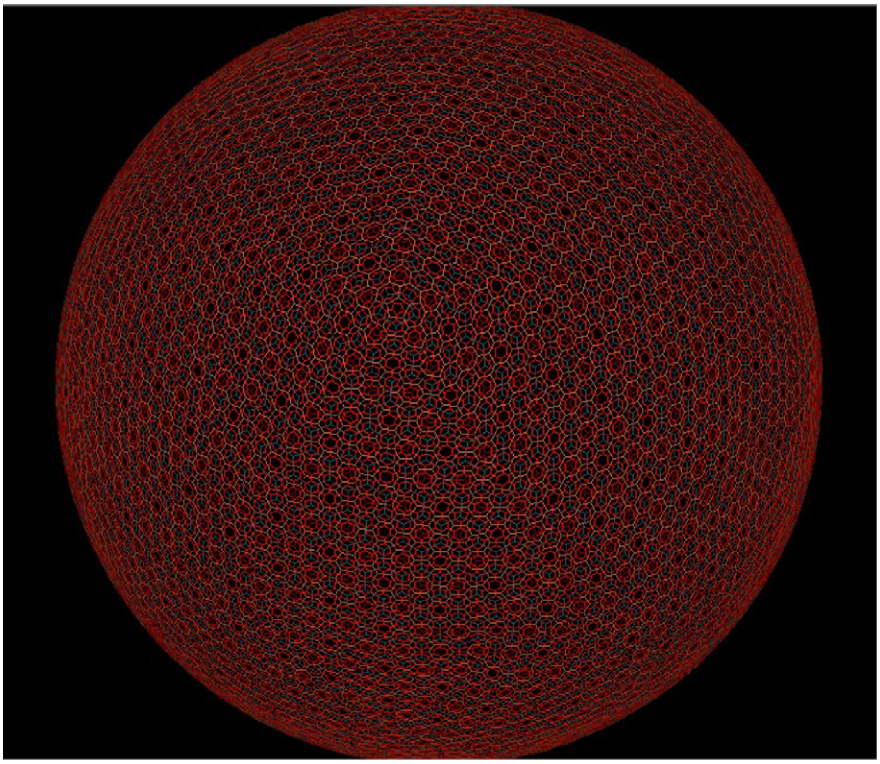

Fig. 1. ISEA3H.

$$|a| + |b| + |c| = 3^{\frac{n}{2}} \qquad \text{if } n \text{ is even}$$

$$\left.\begin{array}{l} |a| + |b| + |c| = 3^{\frac{n+1}{2}} \\ a \equiv b \equiv c \bmod 3 \end{array}\right. \qquad \text{if } n \text{ is odd.}$$

Then the A3-coordinates are, in turn, encoded by a string of $n+3$ digits, each digit an element of the set $\{-1, 0, 1\}$.

Each hexagonal cell has six immediate neighbors. Moreover, OA3HDGG has a treelike structure, each hexagonal cell at level n having seven children at level $n+1$ and 1 or 3 parents at level $n-1$. The definitions of these and other relevant terms appear in Section 2. Given the index of an arbitrary cell at any level, extremely simple rules are given in Sections 3 and 4 for the following basic procedures that are essential to many global grid applications.

Determine the location of the cell.
Determine the neighbors of the cell.
Determine the children of the cell.
Determine the parents of the cell.
Perform local arithmetic in the vicinity of the cell.

2. A3HDGG

The usual definition of the A3HDGG is recursive. Consider either a polyhedron or a sphere of radius 1 centered at the origin of $\mathbb{R}^3$. The barycenter of a triangle t with vertices $\{x_1, x_2, x_3\}$ on the surface of the polyhedron or on the sphere is given by

$$\beta(t) = \begin{cases} (x_1 + x_2 + x_3)/|x_1 + x_2 + x_3| & \text{if on the sphere,} \\ \frac{1}{3}(x_1 + x_2 + x_3) & \text{if on the polyhedron.} \end{cases}$$

The addition is vector addition, and the norm is the usual Euclidean norm. Dividing by the norm in the first formula insures that the barycenter remains on the surface of the sphere. In the second formula the barycenter remains on the surface of the polyhedron.

By a tessellation T of a polyhedron or of the sphere, we mean a collection of closed, non-overlapping cells that cover the surface. On a polyhedron the edges are straight lines and, on the sphere, arcs of great circles. For a tessellation T the notation $V(T)$ and $E(T)$ is used for the set of vertices and edges, respectively. For a tessellation T let

$$B(T) = \{\beta(t) \mid t \in T\}$$

denote the set of barycenters of its cells. The definition of A3HDGG is based on two basic operations on tessellations.

1. *The dual*. For a tessellation T with vertex set V, the dual tessellation $D(T)$ has vertex set $\beta(T)$. Two vertices of $D(T)$ are joined by an edge if and only if the corresponding cells of T share an edge.
2. *Central subdivision*. The central subdivision $C(T)$ of tessellation T has vertex set $V(T) \cup \beta(T)$. The edge set of $C(T)$ is the union of $E(T)$ and the set of edges formed by joining $\beta(t)$ to each vertex of t for all $t \in T$.

Define a sequence (T_n, H_n) of dual pairs of tessellations on either the sphere or polyhedron as follows. First, T_0 is the polyhedron itself centered at the origin of $\mathbb{R}^3$ or its radial projection onto the surface of the sphere. The sequence is then defined recursively by

$$H_n = D(T_n),$$
$$T_{n+1} = C(H_n).$$

The sequence H_n of tessellations is the one called the A3HDGG. The number n is referred to as the *level of resolution* of A3HDGG. A patch of the dual tessellations T_n and H_n, as well as the central subdivision operation, is shown in Fig. 2. Note that $V_n := V(T_n)$ is the set of cell centers of the tessellation H_n. Fig. 3 shows a patch of two successive subdivisions T_n, T_{n+1} and H_n, H_{n+1}. The following properties are either obvious or easily proved by induction.

1. The tessellation T_n is a triangulation for each n, i.e., the faces are triangles. Moreover, if T_0 is the octahedron, then T_n contains exactly $20 \cdot 3^n$ triangles.
2. The nth level of resolution H_n of OA3HDGG contains exactly $4(3^n - 1)$ hexagons and 6 squares.
3. The sets of barycenters of cells of the A3HDGG are nested: $V_0 \subset V_1 \subset V_2 \subset \cdots$
4. Each hexagonal cell h of H_n intersects exactly seven cells of H_{n+1}, a *central child* (with the same barycenter as h) and six *neighboring children*.

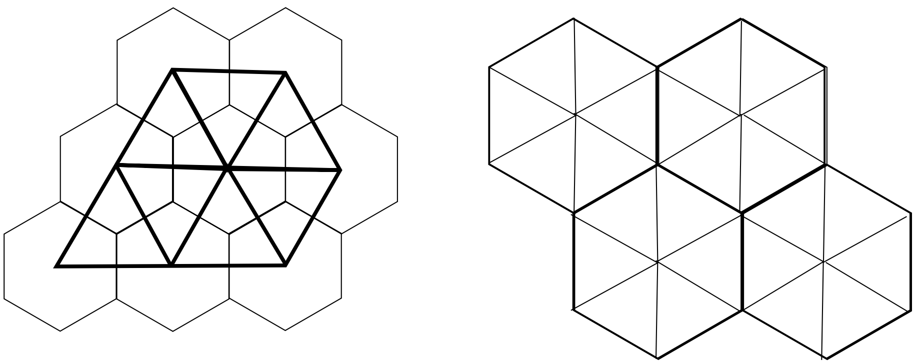

Fig. 2. Basic operations on a tessellation.

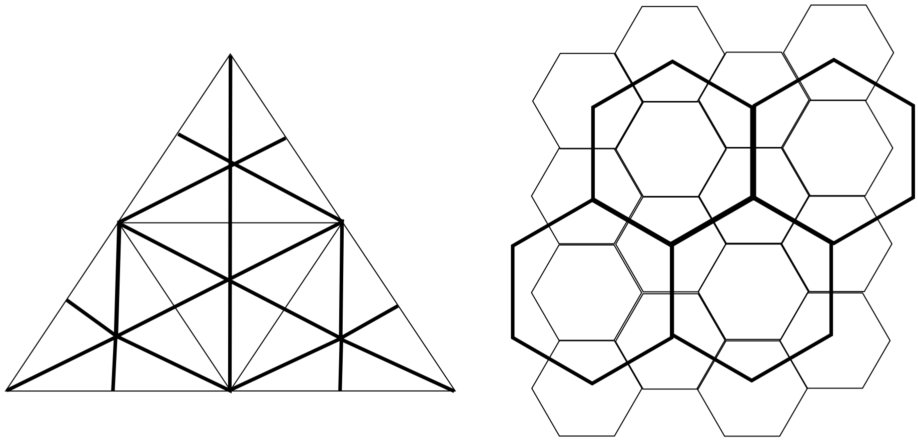

Fig. 3. Two successive subdivisions.

5. Each cell h of H_n intersects either 1 or 3 cells of H_{n-1}, one cell in the case that h is a central child, three cells in the case that h is a neighboring child. These cells at level $n-1$ are called the *parents* of h.

3. OA3HDGG coordinate geometry

Subsequently in this paper, the initial triangulation T_0 is the octahedron with vertices located at $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, $(0, 0, \pm 1)$ in $\mathbb{R}^3$. This section contains fundamental facts about the coordinate geometry of OA3HDGG. As before, let V_n denote the set of vertices of the triangulation T_n, i.e., the set of barycenters of the cells of H_n on the surface of the octahedron. All congruences in this paper are modulo 3. All the propositions below will be proved simultaneously at the end of this section.

Proposition 1. *The set V_n of barycenters of cells of OA3HDGG at resolution n is given by*

$$V_n = \begin{cases} \left\{ \frac{1}{3^{\frac{n}{2}}}(a,b,c) : a,b,c \in \mathbb{Z}, |a|+|b|+|c| = 3^{\frac{n}{2}} \right\} & \text{if } n \text{ is even} \\ \left\{ \frac{1}{3^{\frac{n+1}{2}}}(a,b,c) : a,b,c \in \mathbb{Z}, |a|+|b|+|c| = 3^{\frac{n+1}{2}}, a \equiv b \equiv c \right\} & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 1 gives the Cartesian coordinates of the cell centers of OA3HDGG on the surface of the octahedron. A projection map (the possibilities not discussed here) provides the corresponding point on the sphere. Define the *A3-coordinates* of any vertex of T_n, or the corresponding cell in H_n, as the ordered triple (a, b, c) of integers as given in Proposition 1. In other words the A3-coordinates of a cell with center $(1/3^{\frac{n}{2}})$ (a, b, c), n even, or $(1/3^{\frac{n+1}{2}})$ (a, b, c), n odd, is simply (a, b, c). When no confusion arises, no distinction will be made between a cell and its A3-coordinates.

Proposition 2. *The vertices of T_n with A3-coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) are joined by an edge in T_n if and only if*

$$|a_i - b_i| \leqslant 1, \quad i = 1,2,3 \quad \text{if } n \text{ is even},$$
$$|a_i - b_i| \leqslant 2, \quad i = 1,2,3 \quad \text{if } n \text{ is odd}.$$

A hexagonal cell has six cells immediately adjacent; these are called its *neighbors*. A square cell has four neighbors.

Proposition 3. *The neighbors of a cell in H_n with A3-coordinates (a_1, a_2, a_3) have A3-coordinates (b_1, b_2, b_3), where, for i = 1, 2, 3,*

$$|a_i - b_i| \leqslant \begin{cases} 1 & if\ n\ is\ even, \\ 2 & if\ n\ is\ odd. \end{cases}$$

Using Proposition 3 the six neighbors of a hexagonal cell (a, b, c) can be listed:

$$\begin{matrix} (a+1,b-1,c),(a+1,b,c-1),(a,b+1,c-1), \\ (a-1,b+1,c),(a-1,b,c+1),(a,b-1,c+1), \end{matrix} \quad \text{if } n \text{ is even}$$
$$\begin{matrix} (a+2,b-1,c-1),(a-1,b+2,c-1),(a-1,b-1,c+2), \\ (a-2,b+1,c+1),(a+1,b-2,c+1),(a+1,b+1,c-2) \end{matrix} \quad \text{if } n \text{ is odd.}$$

The ± 1's in the above formula are to be understood subject to the following conventions. The sign changes if the number before it is negative, and, if something is subtracted from 0, then the other signs in the ordered triple change. (See the following examples.)

Example 1. The tessellation H_3 of the octahedron consists of 104 hexagons and 6 squares. The six squares have A3-coordinates $(\pm 9,0,0)$, $(0,\pm 9,0)$, $(0,0,\pm 9)$.

According to Proposition 1, the hexagonal cell with A3-coordinates $(6,3,0) \in H_4$, for example, has Cartesian coordinates $(\frac{2}{3}, \frac{1}{3}, 0)$.

According to Proposition 3, the six neighbors of cell $(6,3,0) \in H_3$ are

$$\begin{matrix} (4,4,1) & (7,1,1) & (5,2,-2) \\ (4,4,-1) & (7,1,-1) & (5,2,2). \end{matrix}$$

The neighbors of the square cell $(9,0,0)$ are $(8,1,0)$, $(8,-1,0)$, $(8,0,-1)$, and $(8,0,-1)$.

Example 2. The tessellation H_4 consists of 320 hexagons and 6 squares. The six squares have coordinates $(\pm 9,0,0)$, $(0,\pm 9,0)$, $(0,0,\pm 9)$.

The hexagonal cell $(6,3,0) \in H_3$ has Cartesian coordinates $(\frac{2}{3}, \frac{1}{3}, 0)$.

The six neighbors of cell $(6,3,0) \in H_4$ are

$$\begin{matrix} (7,2,0) & (6,2,1) & (5,4,0) \\ (5,3,1) & (6,2,-1) & (5,3,-1). \end{matrix}$$

The neighbors of the square cell $(9,0,0)$ are $(7,1,1)$, $(7,1,-1)$, $(7,-1,1)$, $(7,-1,-1)$.

Proposition 4.

1. *The central child in H_{n+1} of $(a, b, c) \in H_n$ is*
$$\begin{cases} 3(a,b,c) & if\ n\ is\ even \\ (a,b,c) & if\ n\ is\ odd. \end{cases}$$
2. *Cell $(a, b, c) \in H_n$ is a central child if and only if*
$$\begin{cases} a \equiv b \equiv c & if\ n\ is\ even \\ a \equiv b \equiv c \equiv 0 & if\ n\ is\ odd. \end{cases}$$
3. *The neighboring children of (a, b, c) are, in either the even or odd case, the neighbors of the central child as given by Proposition 3.*

Proposition 5. *The parent in H_{n-1} of a central child $(a, b, c) \in H_n$ is*

$$\begin{cases} (a,b,c) & if\ n\ is\ even \\ \frac{1}{3}(a,b,c) & if\ n\ is\ odd. \end{cases}$$

Proposition 6. *Let $h = (a, b, c) \in H_n$ be a neighboring child.*

1. *For n even, the cell h has exactly three neighbors (d, e, f) with the property that $d \equiv e \equiv f$. These three are the A3-coordinates of the parents of (a, b, c) in H_{n-1}.*
2. *For n odd, the cell h has exactly three neighbors (d, e, f) with the property that $d \equiv e \equiv f \equiv 0$. For these three the triples $\frac{1}{3}(d, e, f)$ are the A3-coordinates of the parents of (a, b, c).*

Examples 1 and 2 (continued). According to Proposition 4, cell $(5,2,2) \in H_4$ is a central child. According to Proposition 5, the parent of $(5,2,2) \in H_4$ is $(5,2,2) \in H_3$.

According to Proposition 4, cell $(15,6,6) \in H_5$ is a central child. According to Proposition 5, the parent of $(15,6,6)$ is $(5,2,2) \in H_4$.

According to Proposition 4 the central child of $h = (5, 2, 2) \in H_3$ is $(5,2,2) \in H_4$, and the neighboring children in H_4 of h are

$$(6,1,2), \quad (6,2,1), \quad (5,3,1), \quad (5,1,3), \quad (4,3,2), \quad (4,2,3).$$

The central child of $h = (5,2,2) \in H_4$ is $(15,6,6) \in H_5$. The neighboring children in H_5 of h are

$$(17,5,5), \quad (13,7,7), \quad (14,8,5), \quad (16,4,7), \quad (14,5,8), \quad (16,7,4).$$

According to Proposition 4, cell $(5,2,2) \in H_3$ is a neighboring child. According to Proposition 6, the parents in H_2 of $(5,2,2) \in H_3$ are

$$(2,1,0), \quad (2,0,1), \quad (1,1,1).$$

The cell $(4,3,2) \in H_4$ is a neighboring child. The parents in H_3 of $(4,3,2) \in H_4$ are

$$(5,2,2), \quad (3,3,3), \quad (4,4,1).$$

Proof. Proposition 3 follows from Proposition 2, Proposition 5 from Proposition 4, and Proposition 4 part 3 from Proposition 3. The proof of Propositions 1 and 2 is by induction on the level n. For $n = 0$, 1 these are easily checked. Assume that the propositions hold for level n. We will show that they hold for level $n + 1$.

By definition $V_{n+1} = V_n \cup \beta(T_n)$. To prove Proposition 1 it is sufficient to show that the sets on the right-hand sides in the proposition satisfy the same recurrence. Consider the case where n is even first. By Proposition 2 applied to level n, any set of three vertices of a triangle in T_n can be expressed in terms of A3-coordinates as

$$t = \{(a,b,c), (a+1, b-1, c), (a, b-1, c+1)\},$$

for some triple of integers such that $|a| + |b| + |c| = 3^{\frac{n}{2}}$. The ±1's in the above formula are to be understood subject to the conventions stated after Proposition 3. For example $\{(6, 0, -3), (5, -1, -3), (6, -1, -2)\}$ is such a triangle by taking $(a, b, c) = (6, 0, -3)$. The barycenter of the three vertices in t is $\beta(t) = \frac{1}{3} \cdot \frac{1}{3^{n/2}} (3a + 1, 3b - 2, 3c + 1)$. Therefore

$$\beta(T_n) = \left\{ \frac{1}{3^{(n+2)/2}} (a,b,c) : |a| + |b| + |c| = 3^{\frac{n+2}{2}}, \quad a \equiv b \equiv c \not\equiv 0 \right\}.$$

The right-hand side (RHS) in Proposition 1 is given by

$$\begin{aligned} \mathrm{RHS}_{n+1} &= \left\{ \tfrac{1}{3^{(n+2)/2}} (a,b,c) : |a| + |b| + |c| = 3^{\frac{n+2}{2}}, \quad a \equiv b \equiv c \right\}, \\ \mathrm{RHS}_n &= \left\{ \tfrac{1}{3^{n/2}} (a,b,c) : |a| + |b| + |c| = 3^{\frac{n}{2}} \right\}, \\ &= \left\{ \tfrac{1}{3^{(n+2)/2}} (a,b,c) : |a| + |b| + |c| = 3^{\frac{n+2}{2}}, a \equiv b \equiv c \equiv 0 \right\}. \end{aligned}$$

Hence $\mathrm{RHS}_{n+1} = \mathrm{RHS}_n \cup \beta(T_n)$, which proves Proposition 1 in the case that n is even.

Now consider the case where n is odd. By Proposition 2 applied to level n any set of three vertices of a triangle in T_n can be expressed in terms of A3-coordinates as

$$t = \{(a,b,c), (a+1, b+1, c-2), (a-1, b+2, c-1)\},$$

for some triple of integers such that $|a|+|b|+|c| = 3^{\frac{n+1}{2}}$ and $a \equiv b \equiv c$. The ± 1's in the above formula are to be understood subject to the same conventions as in the even case. For example $\{(4,5,0), (3,4,-2), (5,3,-1)\}$ is such a triangle by taking $(a, b, c) = (4,5,0)$. The barycenter of the three vertices in t is $\beta(t) = \frac{1}{3} \cdot \frac{1}{3^{(n+1)/2}}(3a, 3b+3, 3c-3) = \frac{1}{3^{(n+1)/2}}(a, b+1, c-1)$. Therefore

$$\beta(T_n) = \left\{ \frac{1}{3^{(n+1)/2}}(a,b,c) : |a|+|b|+|c| = 3^{\frac{n+1}{2}}, a \not\equiv b \not\equiv c \not\equiv a \right\}.$$

The right-hand side (RHS) in Proposition 1 is given by

$$\begin{aligned} \mathrm{RHS}_{n+1} &= \left\{ \tfrac{1}{3^{(n+1)/2}}(a,b,c) : |a|+|b|+|c| = 3^{\frac{n+1}{2}} \right\} \\ \mathrm{RHS}_n &= \left\{ \tfrac{1}{3^{(n+1)/2}}(a,b,c) : |a|+|b|+|c| = 3^{\frac{n+1}{2}}, a \equiv b \equiv c \right\}. \end{aligned}$$

Hence $\mathrm{RHS}_{n+1} = \mathrm{RHS}_n \cup \beta(T_n)$, which proves Proposition 1 in the case that n is odd.

Concerning Proposition 2, according to the construction given in Section 2 there are two types of edges $\{x, y\}$ at level $n+1$. Either

1. $x \in V_n$, $y = \beta(t)$, where $x \in t \in T_n$, or
2. $x = \beta(t)$, $y = \beta(t')$, where t, t' are neighbors in T_n.

We will prove the type (1) case when n is even; the odd case is similar. By the induction hypothesis applied to Propositions 1 and 2,

$$x = \frac{1}{3^{n/2}}(a,b,c) = \frac{1}{3^{(n+2)/2}}(3a, 3b, 3c)$$

and $t = \{(a, b, c), (a+1, b-1, c), (a, b-1, c+1)\}$ in terms of A3-coordinates. Then

$$y = \beta(t) = \frac{1}{3}\frac{1}{3^{n/2}}(3a+1, 3b-2, 3b+1) = \frac{1}{3^{(n+2)/2}}(3a+1, 3b-2, 3b+1).$$

So the A3-coordinates of x and y differ by at most 2 in absolute value.

We now prove the type (2) case when n is odd; the even case is similar. By the induction hypothesis $t = \{(a, b, c), (a+1, b+1, c-2), (a-1, b+2, c-1)\}$ and $t' = \{(a, b, c), (a+1, b+1, c-2), (a+2, b-1, c-1)\}$ in terms of A3-coordinates. Then

$$x = \beta(t) = \frac{1}{3}\frac{1}{3^{(n+1)/2}}(3a, 3b+3, 3c-3) = \frac{1}{3^{(n+1)/2}}(a, b+1, c-1)$$

and

$$y = \beta(t') = \frac{1}{3}\frac{1}{3^{(n+1)/2}}(3a+3, 3b, 3c-3) = \frac{1}{3^{(n+1)/2}}(a+1, b, c-1).$$

So the A3-coordinates of x and y differ by at most 1 in absolute value.

Concerning Proposition 4, a cell at level n and its central child at level $n+1$ have the same barycenter. Hence, by Proposition 1, that barycenter can be expressed as

$$\begin{aligned} &\tfrac{1}{3^{n/2}}(a,b,c) = \tfrac{1}{3^{(n+1+1)/2)}}(3a, 3b, 3c) && \text{if } n \text{ is even,} \\ &3^{\frac{n+1}{2}}(a,b,c), a \equiv b \equiv c && \text{if } n \text{ is odd.} \end{aligned}$$

This proves parts 1 and 2 of Proposition 4.

Consider the n even case in Proposition 6. The cell $(a, b, c) \in H_n$ is a neighboring child, according to Proposition 4, if and only if a, b and c lie in different congruence classes modulo 3. Without loss of generality assume that $(a, b, c) \equiv (0,1,2)$, i.e., $a \equiv 0$, $b \equiv 1$, $c \equiv 2$. Modulo 3 the coordinates of its neighbors, according to Proposition 3, are $(0,0,0)$, $(1,1,1)$, $(2,2,2)$, $(0,2,1)$, $(2,1,0)$, $(1,0,2)$. The first three are of the form (d, e, f)

where $d \equiv e \equiv f$. These are, by Proposition 4, central children. The A3-coordinates of the parent of each such (d, e, f) is, according to Proposition 5, again (d, e, f). Since each such $(d, e, f) \in H_n$ is a neighbor of (a, b, c), the parent $(d, e, f) \in H_{n-1}$ overlaps cell (a, b, c) according to the construction in Section 2. Hence each such cell $(d, e, f) \in H_{n-1}$ is a parent of (a, b, c).

In the case that n is odd, $(a, b, c) \in H_n$ is a neighboring child, according to Proposition 4, if and only if $a \equiv b \equiv c \not\equiv 0$. Then $(a, b, c) \equiv (\pm 1, \pm 1, \pm 1)$, i.e., $a \equiv \pm 1$, $b \equiv \pm 1$, $c \equiv \pm 1$), the signs all the same. Modulo 3 the coordinates of its neighbors, according to Proposition 3, are $(0,0,0)$, $(0,0,0)$, $(0,0,0)$, $(\pm 1, \pm 1, \pm 1)$, $(\pm 1, \pm 1, \pm 1)$, $(\pm 1, \pm 1, \pm 1)$. The first three are of the form (d, e, f) where $d \equiv e \equiv f \equiv 0$. These are, by Proposition 4, central children. The parent of each such (d, e, f) is, according to Proposition 5, $\frac{1}{3}(d,e,f)$. Since each such $(d, e, f) \in H_n$ is a neighbor of (a, b, c), the parent $\frac{1}{3}(d,e,f) \in H_{n-1}$ by definition overlaps cell (a, b, c). Hence each such cell $\frac{1}{3}(d,e,f) \in H_{n-1}$ is a parent of (a, b, c). □

4. Local arithmetic

Using the A3-coordinates, a local vector arithmetic can be efficiently implemented. By "local" we mean centered at any cell of OA3HDGG and restricted to a single face of the octahedron. By "vector arithmetic" we mean usual vector addition and multiplication by scalars for vectors contained on a single face of the octahedron.

The *octant* of a triple (a, b, c) of integers is the ordered triple of signs (+ or −) of the three entries. The octant of $(5, -2, 2)$, for example, is $(+, -, +)$. Two vertices of the triangulation T_n lie on the same face of the octahedron if and only if their A3-coordinates are in the same octant. Let $\mathbf{x}_0$ be a fixed cell $h_0 \in H_n$ given by its A3-coordinates. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the A3-coordinates of two other cells $h_1 \in H_n$ and $h_2 \in H_n$ in the same octant as $\mathbf{x}_0$. Let $\mathbf{v}_1$ denote the vector pointing from the center of h_0 to the center of h_1; similarly $\mathbf{v}_2$ the vector pointing from the center of h_0 to h_2.

Proposition 7.

1. *The vector sum* $\mathbf{v}_1 + \mathbf{v}_2$ *points from* $\mathbf{x}_0$ *to*

$$\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_0,$$

   where the sum of the $\mathbf{x}_i$ *in the above formula is the usual addition in* $\mathbb{R}^3$. *The formula is valid as long as* $\mathbf{x}_1 + \mathbf{x}_2 - \mathbf{x}_0$ *lies in the same octant as* $\mathbf{x}_0$.
2. *For an integer* k, *the scalar product* $k\mathbf{v}_1$ *points from* $\mathbf{x}_0$ *to*

$$k\mathbf{x}_1 + (1-k)\mathbf{x}_0.$$

Example. Consider the vector $\mathbf{v}_1$ pointing from $(9,9,9) \in H_6$ to $(12,5,10) \in H_6$ and the vector $\mathbf{v}_2$ pointing from $(9,9,9) \in H_6$ to $(8,13,6) \in H_6$. According to Proposition 7, $\mathbf{v}_1 + \mathbf{v}_2$ is a vector from $(9,9,9)$ to

$$(12,5,10) + (8,13,6) - (9,9,9) = (11,9,7).$$

The vector $2\mathbf{v}_1$ points from $(9,9,9)$ to

$$2(12,5,10) - (9,9,9) = (15,1,11).$$

5. Indexing OA3HDGG using the balanced ternary

The *balanced ternary* is a base 3 positional number system using the digit set $D = \{-1, 0, 1\}$, with D often referred to as the set of *trits*. Relevant to our application are the following properties of the balanced ternary. Further information on the balanced ternary can be found in Knuth's "The Art of Computer Programming" [9].

1. Every integer, positive or negative, has a unique representation in the balanced ternary. Moreover, every integer between $-3^n/2$ and $3^n/2$ has a unique representation of the form

$$\sum_{k=0}^{n-1} d_k 3^k,$$

where $d_k \in D$.

2. The negative of an integer in balanced ternary is obtained by merely changing the sign of each digit.
3. Two integers in balanced ternary are congruent modulo 3 if and only if they have the same last digit. In particular, an integer is divisible by 3 if and only if the last digit in the balanced ternary is 0.

Example. The integer $(1)(-1)(-1) = 9 - 3 - 1 = 5$ and $(-1)(1)(1) = -5$. The integer $(1)(-1)(0)(0)(1)(-1)(0)$ is divisible by 3.

In the previous section it was shown that the Cartesian coordinates of the barycenter of each OA3HDGG cell can be represented in terms of an ordered triple of integers called the A3-coordinates. An indexing scheme for H_n is now introduced for referencing the cells at any level n using a string of $n+3$ trits. The basic operations detailed in Propositions 1–7 can then be performed base 3 with the digits D. Such a string S of trits encodes the A3-coordinates (a, b, c) as follows. We consider the cases n even and odd separately.

Even $n = 2k$. The first $k+1$ trits in S represent the integer a.

The second $k+1$ trits represent the integer b.

The third integer is given by the formula $c = \pm(3^k - |a| - |b|)$, where the ± 1 is the last trit in S.

Example. If $n = 4$ and $S = (1)(-1)(-1)(0)(1)(0)(-1)$, then

$$\begin{aligned}
a &= (1)(-1)(-1) = 5,\\
b &= (0)(1)(0) = 3,\\
c &= (-1)(9-5-3) = -1,\\
(a,b,c) &= (5,3,-1) \in H_4.
\end{aligned}$$

The basic operations are more efficiently carried out without converting from balanced ternary:

$$\begin{aligned}
a &= (1)(-1)(-1),\\
b &= (0)(1)(0),\\
c &= -[(1)(0)(0) + (-1)(1)(1) + (0)(-1)(0)] = (0)(0)(-1).
\end{aligned}$$

In balanced ternary notation the location of the center of the cell encoded by the string $S = (1)(-1)(-1)(0)(1)(0)(-1)$ is, according to Proposition 1,

$$((1).(-1)(-1), (0).(1)(0), (0).(0)(-1)).$$

Odd $n = 2k - 1$. The first $k+1$ trits in S represent the integer a.

The second $k+1$ trits represent the integer b.

The third integer is given by the formula $c = \pm(3^k - |a| - |b|)$, where the $\pm$ is chosen to make $a \equiv b \equiv c \pmod 3$.

Example. If $n = 3$ and $S = (1)(-1)(-1)(0)(1)(-1)$, then

$$\begin{aligned}
a &= (1)(-1)(-1) = 5,\\
b &= (0)(1)(-1) = 2,\\
c &= \pm(9-5-2) = \pm 2 = 2,\\
(a,b,c) &= (5,2,2) \in H_3.
\end{aligned}$$

To summarize:

- The location of any of the $4 \cdot 3^n + 2$ cells at the nth level of resolution of OA3HDGG is uniquely encoded by its A3-coordinates or as a string of $n+3$ trits.

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