THE ATTRACTOR OF AN ITERATED FUNCTION SYSTEM - RECENT RESULTS

Andrew Vince

CHAOS 2019
Iterated Function System

Iterated function system: \( \{X; f_1, f_2, \ldots, f_N\} \)

\[ f : X \rightarrow X \quad \text{continuous for all} \quad f \in F \]

Hutchinson operator: \( F : \mathbb{H}(X) \rightarrow \mathbb{H}(X) \)

\[ F(B) = \bigcup_{f \in F} f(B) \]

Attractor \( A \in \mathbb{H}(X) \)

- \( F(A) = A \)
- There is an open set \( U \) such that \( A \subset U \subset X \) and for any compact set \( B \subset U \):
  \[ A = \lim_{k \to \infty} F^k(B). \]
\[ F(A) = \bigcup_{f \in F} f(A) = A \]

\[ \lim_{k \to \infty} F^k(B) = A \]
Topics

- When does an IFS have an attractor?
- Transition phenomena
- Fractal transformations
- IFS tilings
When does an IFS have an attractor?

Theorem (Hutchinson 1981)

If each function in an IFS is a contraction, then the IFS has a unique attractor. Moreover, the basin of attraction is $\mathbb{X}$.
**When does an IFS have an attractor?**

**Theorem** (Hutchinson 1981)

If each function in an IFS is a contraction, then the IFS has a unique attractor. Moreover, the basin of attraction is $\mathbb{X}$.

- What is the role of contractivity?
- Does a converse of Hutchinson’s Theorem hold?
affine IFS: \( \mathbb{R}^n \) \( f(x) = Lx + a \)

Möbius IFS: \( \hat{\mathbb{C}} \) \( f(z) = \frac{az + b}{cz + d} \)

projective IFS: \( \mathbb{P}^n := \mathbb{R}^{n+1} \setminus \{0\}/\sim \) \( f(x) = Lx \)

where \( (x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n) \) for all nonzero \( \lambda \in \mathbb{R} \).
Attractor and repeller of a projective plane IFS:
Attractors of Möbius IFSs:
An IFS on a complete metric space $\mathbb{X}$ is called **contractive** if there is a metric $d$ on a nonempty open set $U \subseteq \mathbb{X}$ giving the same topology as the original metric on $\mathbb{X}$ and such that each function in the IFS is a contraction on $U$ with respect to $d$. 
**Theorem**

1. An affine IFS has an attractor if and only if it is contractive on $\mathbb{R}^n$.

2. A Möbius IFS has an attractor $A \neq \mathbb{C}$ if and only if it is contractive on some nonempty open proper subset of $\hat{\mathbb{C}}$.

3. A projective IFS has an attractor avoiding a hyperplane if and only if it is contractive on the closure of some nonempty open set.
Minkowski metric (affine case)

If $K$ is a convex body, then $C := K - K$ is a centrally symmetric convex body. Let

$$d(x, y) = \|x - y\|_C \quad \text{where} \quad \|x\|_C = \inf\{\lambda \geq 0 \mid x \in \lambda C\}$$
\[ \|x\|_C = \inf\{\lambda \geq 0 \mid x \in \lambda C\} \]
\[ \|x\|_C = \inf\{\lambda \geq 0 \mid x \in \lambda C\} \]
Hilbert metric (projective case)

For a convex body $K \subset \mathbb{P}^n$ let

$$d_K(x, y) := \log R(a, b, x, y) = \log \left( \frac{|ay| |bx|}{|ax| |by|} \right).$$
(Möbius case)

\[ d_U(x, y) = \max_{z \notin U} \log \frac{|z - x|}{|z - y|} + \max_{z \notin U} \log \frac{|z - y|}{|z - x|} \]
Phase Transition

joint spectral radius

\[ \sigma = i_1 i_2 \cdots i_k \quad \quad L_\sigma = L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_k} \]

\[ \rho_k = \sup_{\sigma} \rho(L_\sigma) \quad \quad \rho = \lim_{k \to \infty} (\rho_k)^{1/k} \]

**Theorem**  A compact affine IFS \( F \) on \( \mathbb{R}^n \) has an attractor if and only if \( \rho(F) < 1 \). If \( \rho(F) > 1 \), then no nonempty bounded set \( A \) exists such that \( F(A) = A \).
Phase Transition

Linear Case:

If $\rho(F) < 1$, then the attractor is a single point; if $\rho(F) > 1$, then there is no attractor.

**Theorem** An irreducible linear IFS $\mathcal{F}$ with $\rho(\mathcal{F}) = 1$ has a compact invariant set that is centrally symmetric and star-shaped.
Phase Transition

Andrew Vince

The Attractor of an Iterated Function System - Recent Results
Fractal Transformation

$[N] = \{1, 2, 3, \ldots, N\}$

code space $\mathbb{I} = \{\sigma = \sigma_1 \sigma_2 \sigma_3 \cdots : \sigma_n \in [N] \text{ for all } n\}$

The coding map $\pi : \mathbb{I} \rightarrow A$

$$\pi(\sigma) = \lim_{k \rightarrow \infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(B)$$
section:

\( \pi : \mathbb{I} \rightarrow A \)

\( \tau : A \rightarrow \mathbb{I} \)

\[ \pi \circ \tau = \text{id} \]
section:

\[ \pi : \mathbb{I} \to A \]
\[ \tau : A \to \mathbb{I} \]

\[ \pi \circ \tau = \text{id} \]

\[ F = \{ f_1, f_2, \ldots, f_N \} \]
\[ G = \{ g_1, g_2, \ldots, g_N \} \]

\[ T_{FG} : A_F \to A_G \]
\[ T_{FG} = \pi_G \circ \tau_F \]
Example
3-D Example
Hilbert Space Filling Curve

\[ F = \left\{ \mathbb{R}; \quad f_i(x) = \frac{x + i - 1}{4}, \quad i = 1, 2, 3, 4 \right\} \]

\[ G = \left\{ \mathbb{R}^2; \quad g_i, \quad i = 1, 2, 3, 4 \right\} \]

Andrew Vince

The Attractor of an Iterated Function System - Recent Results
Area Preserving Fractal Homeomorphism

The Attractor of an Iterated Function System - Recent Results
Global Area Preserving Fractal Homeomorphisms
Tilings from a Graph IFS

Archimedean tilings

4:4:4:4 p4m
4:8:8 p4m
3:3:3:4 cmm
3:3:3:4 p4g
3:3:3:3 p6m
3:6:36 p6m
3:12:12 p6m
6:6:6 p6m
3:4:6:4 p6m
4:6:12 p6m
3:3:3:6 p6m
• **non-periodic:** There is no translational symmetry.
- **non-periodic**: There is no translational symmetry.

- **repetitive**:

  For every finite patch $P$, there is $R > 0$ such that a copy of $P$ appears in every disk of radius $R$. 
• **non-periodic:** There is no translational symmetry.

• **repetitive:**
  
  For every finite patch $P$, there is $R > 0$ such that a copy of $P$ appears in every disk of radius $R$.

• **self-similar:**
  
  There is a similarity transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for every $t \in T$, the larger tile $\phi(t)$ is in turn tiled by tiles in $T$. 
Penrose tiling
A **self-similar tiling** is a tiling of the plane with the properties:

1. finitely many tiles up to congruence
2. quasiperiodic
3. self-similar
A self-similar tiling is a tiling of the plane with the properties:

1. finitely many tiles up to congruence
2. quasiperiodic
3. self-similar
Graph Iterated Function System

A parameter is a reverse infinite path in the graph.
Graph Iterated Function System

A parameter is a reverse infinite path in the graph.

**Theorem.** For infinitely many parameters $P$, the tiling $T(P)$ is a self-similar tiling.
Andrew Vince
The Attractor of an Iterated Function System - Recent Results
Andrew Vince
The Attractor of an Iterated Function System - Recent Results