The Attractor of an Iterated Function System - Recent Results

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CHAOS 2019

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Iterated Function System

Iterated function system: $\{\mathbb{X}; f_1, f_2, \ldots, f_N\}$

$$f : \mathbb{X} \to \mathbb{X} \quad \text{continuous for all} \quad f \in F$$

Hutchinson operator: $F : \mathbb{H}(\mathbb{X}) \to \mathbb{H}(\mathbb{X})$

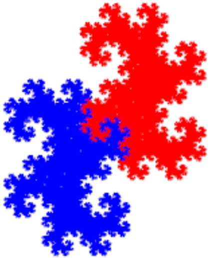
$$F(B) = \bigcup_{f \in F} f(B)$$

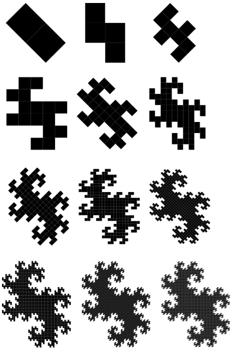
Attractor $A \in \mathbb{H}(\mathbb{X})$

- $F(A) = A$
- There is an open set U such that $A \subset U \subset \mathbb{X}$ and for any compact set $B \subset U$:

$$A = \lim_{k \to \infty} F^k(B).$$

$$F(A) = \bigcup_{f \in F} f(A) = A \qquad \lim_{k \to \infty} F^k(B) = A$$





Topics

- When does an IFS have an attractor?
- Transition phenomena
- Fractal transformations
- IFS tilings

When does an IFS have an attractor?

Theorem (Hutchinson 1981)

If each function in an IFS is a contraction, then the IFS has a unique attractor. Moreover, the basin of attraction is $\mathbb{X}$.

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- What is the role of contractivity?
- Does a converse of Hutchinson's Theorem hold?

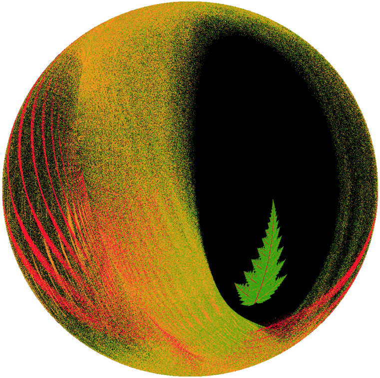
affine IFS: $\mathbb{R}^n$ $f(x) = Lx + a$

Möbius IFS: $\widehat{\mathbb{C}}$ $f(z) = \frac{az+b}{cz+d}$

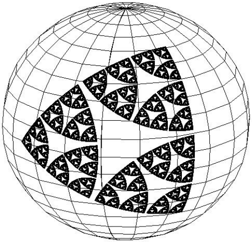
projective IFS: $\mathbb{P}^n := \mathbb{R}^{n+1} \setminus \{0\} / \sim$ $f(x) = Lx$

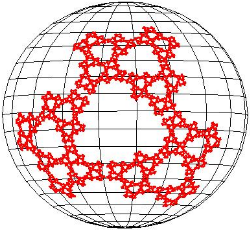
where $(x_0, \ldots, x_n) \sim (\lambda x_0, \ldots, \lambda x_n)$ for all nonzero $\lambda \in \mathbb{R}$.

Attractor and repeller of a projective plane IFS:



Attractors of Möbius IFSs:





An IFS on a complete metric space $\mathbb{X}$ is called contractive if there is a metric d on a nonempty open set $U \subseteq \mathbb{X}$ giving the same topology as the original metric on $\mathbb{X}$ and such that each function in the IFS is a contraction on U with respect to d.

Theorem

1. An affine IFS has an attractor if and only if it is contractive on $\mathbb{R}^n$.
2. A Möbius IFS has an attractor $A \neq \mathbb{C}$ if and only if it is contractive on some nonempty open proper subset of $\widehat{\mathbb{C}}$.
3. A projective IFS has an attractor avoiding a hyperplane if and only if it is contractive on the closure of some some nonempty open set.

Minkowski metric (affine case)

If K is a convex body, then $C := K - K$ is a centrally symmetric convex body. Let

$$d(x,y) = \|x-y\|_C \qquad \text{where} \qquad \|x\|_C = \inf\{\lambda \geq 0 \,|\, x \in \lambda C\}$$

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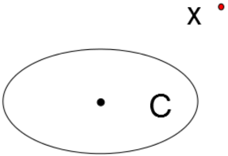
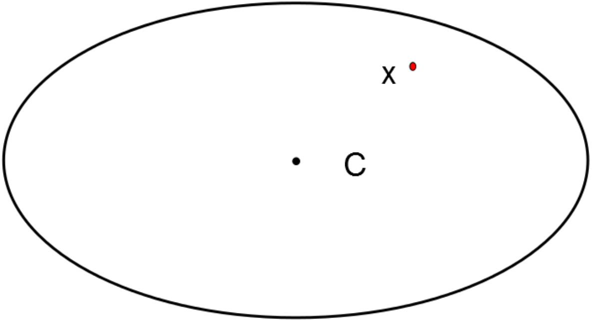
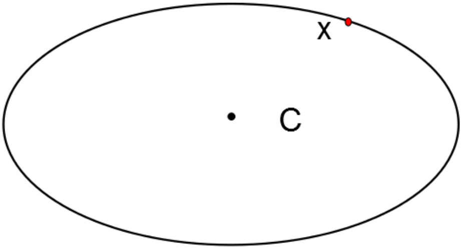




$$\|x\|_C = \inf\{\lambda \geq 0 \,|\, x \in \lambda C\}$$

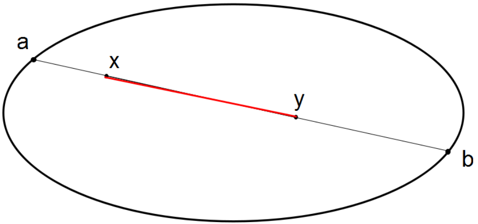


$$\|x\|_C = \inf\{\lambda \geq 0 \,|\, x \in \lambda C\}$$

Hilbert metric (projective case)

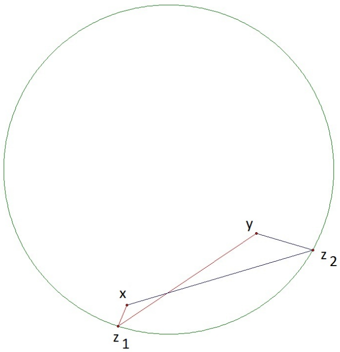
For a convex body $K \subset \mathbb{P}^n$ let

$$d_K(x,y) := \log R(a,b,x,y) = \log \left(\frac{|ay|\,|bx|}{|ax|\,|by|} \right).$$



(Möbius case)

$$d_U(x,y) = \max_{z \notin U} \log \frac{|z-x|}{|z-y|} + \max_{z \notin U} \log \frac{|z-y|}{|z-x|}$$



PHASE TRANSITION

joint spectral radius

$$\sigma = i_1 i_2 \cdots i_k \qquad L_\sigma = L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_k}$$

$$\rho_k = \sup_\sigma \rho(L_\sigma) \qquad \rho = \lim_{k\to\infty} (\rho_k)^{1/k}$$

Theorem A compact affine IFS $\mathcal{F}$ on $\mathbb{R}^n$ has an attractor if and only if $\rho(\mathcal{F}) < 1$. If $\rho(\mathcal{F}) > 1$, then no nonempty bounded set A exists such that $F(A) = A$.

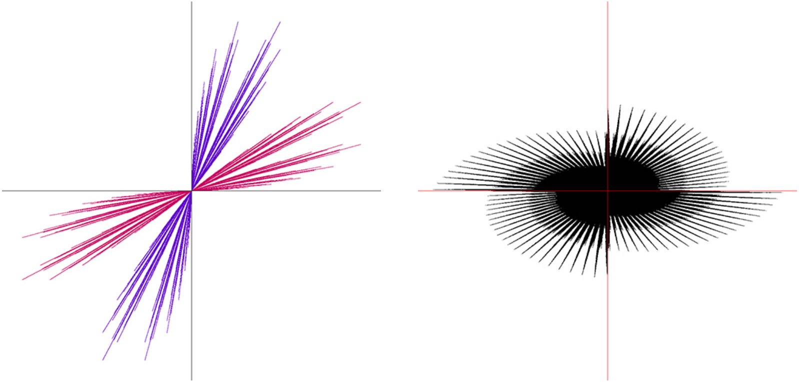
Phase Transition

Linear Case:

If $\rho(F) < 1$, then the attractor is a single point; if $\rho(\mathcal{F}) > 1$, then there is no attractor.

Theorem An irreducible linear IFS $\mathcal{F}$ with $\rho(\mathcal{F}) = 1$ has a compact invariant set that is centrally symmetric and star-shaped.

Phase Transition



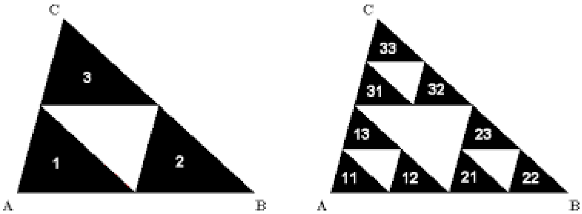
Fractal Transformation

$[N] = \{1, 2, 3, \dots, N\}$

code space $\quad \mathbb{I} = \{\sigma = \sigma_1\sigma_2\sigma_3\cdots : \sigma_n \in [N] \text{ for all } n\}$

The coding map $\quad \pi : \mathbb{I} \to A$

$$\pi(\sigma) = \lim_{k\to\infty} f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_k}(B)$$



section:

$$\pi : \mathbb{I} \to A$$

$$\tau : A \to \mathbb{I}$$

$$\pi \circ \tau = \mathrm{id}$$

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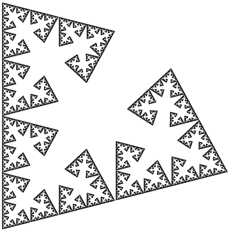
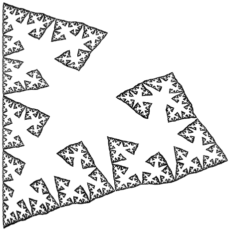
$$F = \{f_1, f_2, ..., f_N\} \qquad G = \{g_1, g_2, ..., g_N\}$$

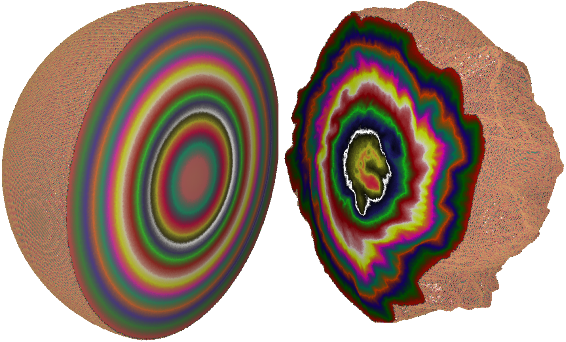
$$T_{FG} : A_F \to A_G$$
$$T_{FG} = \pi_G \circ \tau_F$$

Example





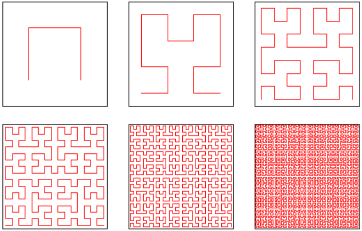
3-D Example

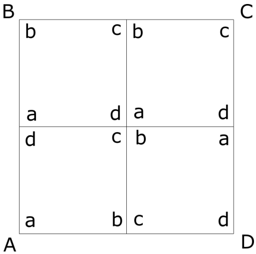


Hilbert Space Filling Curve

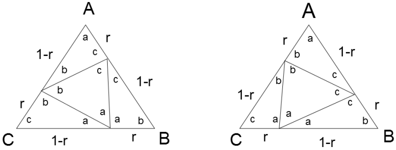
$$F = \left\{ \mathbb{R};\ f_i(x) = \frac{x+i-1}{4},\ i = 1,2,3,4 \right\}$$

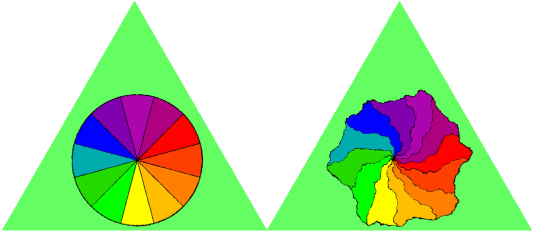
$$G = \left\{ \mathbb{R}^2;\ g_i,\ i = 1,2,3,4 \right\}$$



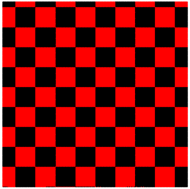


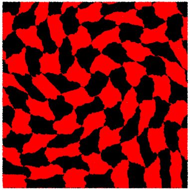
Area Preserving Fractal Homeomorphism

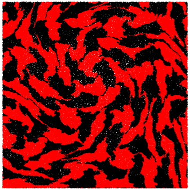




Global Area Preserving Fractal Homeomorphisms

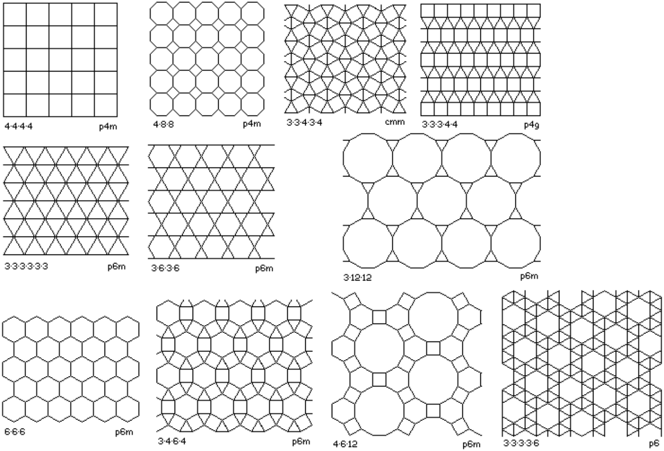


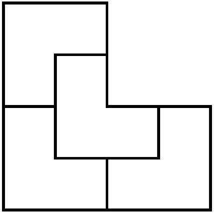


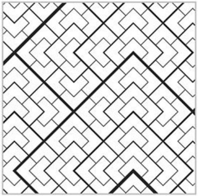


Tilings from a Graph IFS

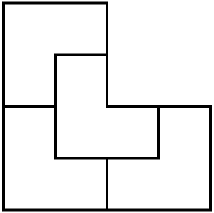
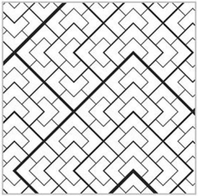
Archimedean tilings







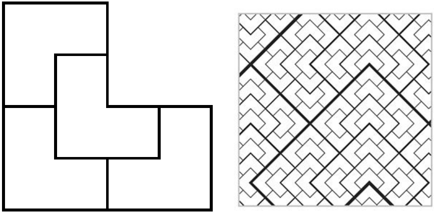
- non-periodic: There is no translational symmetry.

- non-periodic: There is no translational symmetry.

- repetitive:

  For every finite patch P, there is $R > 0$ such that a copy of P appears in every disk of radius R.



- non-periodic: There is no translational symmetry.

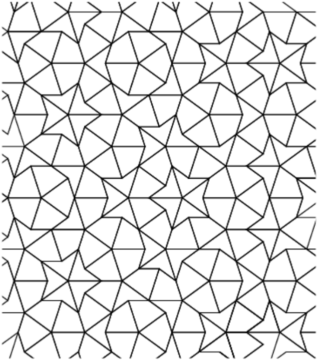
- repetitive:

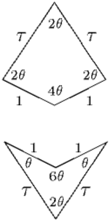
  For every finite patch P, there is $R > 0$ such that a copy of P appears in every disk of radius R.

- self-similar:

  There is a similarity transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that, for for every $t \in T$, the larger tile $\phi(t)$ is in turn tiled by tiles in T.

Penrose tiling





A self-similar tiling is a tiling of the plane with the properties:

1. finitely many tiles up to congruence
2. quasiperiodic
3. self-similar

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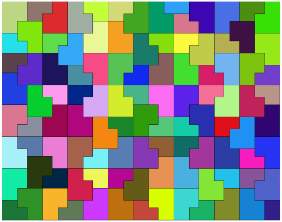
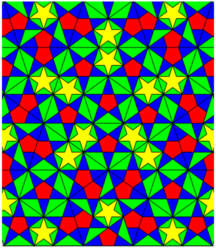
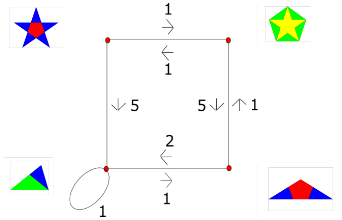
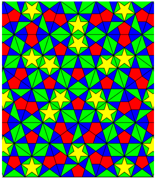




Graph Iterated Function System





A parameter is a reverse infinite path in the graph.

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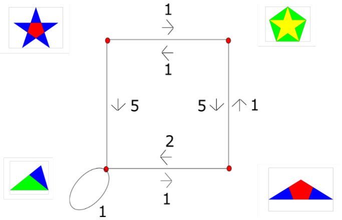

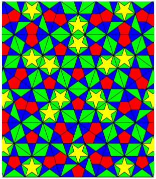

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Theorem. For infinitely many parameters P, the tiling $T(P)$ is a self-similar tiling.