Star Chromatic Number

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ABSTRACT

A generalization of the chromatic number of a graph is introduced such that the colors are integers modulo *n*, and the colors on adjacent vertices are required to be as far apart as possible.

1. INTRODUCTION

The *chromatic number* $\chi(G)$ of a graph is the least number of colors required for a proper vertex coloring of G. A generalization $\chi_k(G)$, $k = 1, 2, \ldots$ of the chromatic number will be given such that the ordinary chromatic number $\chi(G)$ is one of the $\chi_k(G)$. In an ordinary coloring, colors $1, 2, \ldots, k$ are assigned to the vertices so that the colors on adjacent vertices are at least 1 unit apart. In $\chi_k(G)$, which is formally defined in Section 2, colors $1, 2, \ldots, k$ are assigned so that the colors on adjacent vertices are as far apart as possible. A new invariant $\chi^\star(G) = \inf_{k\geq 1}\chi_k(G)$ can then be thought of as the "best possible" coloring. Consider an n-cycle C_n, for example. Whereas the chromatic number of an odd cyle is 3, intuitively it should be "almost 2." In fact, $\chi^\star(C_{2n+1}) = 2 + (1/n)$. Variations on the chromatic number have previously been introduced [1–3], but this natural generalization seems not to have been investigated.

Let G be a connected graph with $n \geq 2$ vertices. It is not surprising that $\chi^\star(G) = \lim_{k\to\infty}\chi_k(G)$ (Corollary 2), but in Section 2 it is shown that in computing $\chi^\star$, only the first n numbers $\chi_1, \ldots, \chi_n$ need be considered: $\chi^\star(G) = \min_{1\leq k\leq n}\chi_k(G)$. Basic properties of $\chi^\star$ are given in Section 3, including its relationship to the ordinary chromatic number and the clique number. It is shown that $\chi^\star(G) \geq 2$ with equality if and only if G is bipartite. Furthermore, if q is any rational number greater than or equal to 2, then a graph G_q is constructed for which $\chi^\star(G_q) = q$.

2. *k*-CHROMATIC NUMBERS

Let Z_k denote the set of integers modulo k, and for any real number x, let $|x|_k$ denote the circular norm of x, i.e., the distance from x to the nearest multiple of k. For example, $|2|_5 = 2 = |3|_5$. Throughout this paper G will be a finite con-

Journal of Graph Theory, Vol. 12, No. 4, 551–559 (1988)
 CCC 0364-9024/88/040551-09$04.00

nected graph with vertex set V. A Z_k*-coloring* of G is a function $c: V \rightarrow Z_k$, where Z_k is the set of integers modulo k. For a Z_k-coloring c let

$$\psi(c) = \frac{k}{\min_{u \text{ adj } v} |c(u) - c(v)|_k} \geq \frac{k}{\lfloor k/2 \rfloor} \geq 2 .$$

If the denominator is 0, then set $\psi(c) = \infty$. The denominator is the least distance between adjacent vertices of G. Hence $\psi(c)$ is the number of colors used per unit color separation at adjacent vertices. Let $C_k = C_k(G)$ denote the set of all Z_k-colorings of G. Define the k-*chromatic number* $\chi_k(G)$ as the least possible $\psi(c)$:

$$\chi_k(G) = \min_{c \in C_k} \psi(c) = \frac{k}{\max_{c \in C_k} \min_{u \text{ adj } v} |c(u) - c(v)|_k} \geq \frac{k}{\lfloor k/2 \rfloor} \geq 2 .$$

In an ordinary coloring, we ask that colors on adjacent vertices be distinct, i.e., one unit apart. In a Z_k-coloring, we ask for more—that the colors on adjacent vertices be as far apart as possible.

Example 1. Consider the 5-cycle. The first few k-chromatic numbers are $\chi_1 = \infty$, $\chi_2 = \infty$, $\chi_3 = 3$, $\chi_4 = 3$, $\chi_5 = 2\frac{1}{2}$. It is shown below (Theorem 2) that the k-chromatic number is never less than $2\frac{1}{2}$. The ordinary chromatic number is 3.

Example 2. Consider the graph G_0 in Figure 1. The first few k-chromatic numbers are $\chi_1 = \infty$, $\chi_2 = \infty$, $\chi_3 = \infty$, $\chi_4 = 4$, $\chi_5 = 5$, $\chi_6 = 6$, $\chi_7 = 3\frac{1}{2}$. The values of the 1-chromatic through 4-chromatic numbers in this example are made clear in Theorem 1. The "best" coloring in the list is the Z_7-coloring, giving a 7-chromatic number of $3\frac{1}{2}$. Theorem 2 again implies that this is actually the least k-chromatic number for any value of k. Notice that the value $3\frac{1}{2}$ is less than the

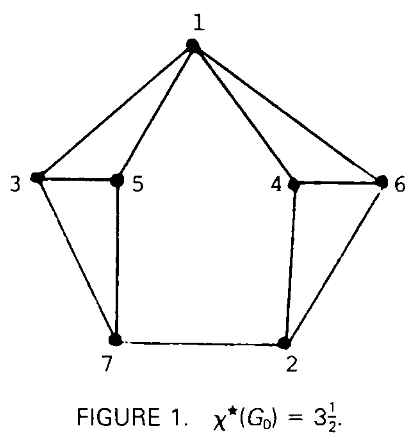

FIGURE 1. $\chi^*(G_0) = 3\frac{1}{2}$.

ordinary chromatic number 4. A referee has pointed out that the Z_7-coloring given in Figure 1 is unique up to automorphisms of the graph and addition of a constant to each vertex (modulo 7).

The first result implies that the k-chromatic numbers generalize the ordinary chromatic number χ in the sense that χ is one of the χ_k.

Theorem 1. If $k = \chi(G)$ then $\chi_k(G) = \chi(G)$. If $k < \chi(G)$ then $\chi_k(G) = \infty$.

Proof. If $\chi(G) = k$ there is an ordinary coloring c: $V \rightarrow Z_k$ such that adjacent vertices are different colors. Then $\min|c(u) - c(v)|_k \geq 1$. Therefore $\chi_k(G) \leq k = \chi(G)$. To prove equality, assume the contrary: $k/\min|c(u) - c(v)|_k = \chi_k(G) < k$ for some Z_k-coloring c. Then $\min|c(u) - c(v)|_k \geq 2$ for all adjacent vertices u and v. Consider the Z_{k-1}-coloring c', where $c'(u) = c(u)$ if $c(u) \neq 0$ and $c'(u) = 1$ if $c(u) = 0$. Then c' is a proper k-1 coloring of G, contradicting $\chi(G) = k$.

If $k < \chi(G)$, then for any coloring c: $V \rightarrow Z_k$ there are two vertices u and v such that $c(u) = c(v)$. Hence $\min|c(u) - c(v)|_k = 0$ and $\chi_k(G) = \infty$. ∎

Theorem 2. For a graph G we have $\chi_k(G) + 1 \geq \chi_{k+1}(G)$.

Proof. Let $c \in C_k$ be such that $\psi(c) = \chi_k(G) \neq \infty$. Define c': $V \rightarrow Z_{k+1}$ by $c'(u) = c(u)$ for all $u \in V$. Then $\min_{u \operatorname{adj} v}|c'(u) - c'(v)|_{k+1} \geq \min_{u \operatorname{adj} v}|c(u) - c(v)|_k \geq 1$. Therefore $\chi_k(G) = \psi(c) = k/\min_{u \operatorname{adj} v}|c(u) - c(v)|_k \geq k/\min_{u \operatorname{adj} v}|c'(u) - c'(v)|_{k+1}$; and $\chi_k(G) + 1 \geq (k + \min_{u \operatorname{adj} v}|c'(u) - c'(v)|_{k+1})/\min_{u \operatorname{adj} v}|c'(u) - c'(v)|_{k+1} \geq (k + 1)/\min_{u \operatorname{adj} v}|c'(u) - c'(v)|_{k+1} = \psi(c') \geq \chi_{k+1}(G)$. ∎

The next result is somewhat surprising. It states that, for a graph with n vertices, the least k-chromatic number occurs for $k \leq n$. The proof relies on real valued functions.

Theorem 3. Let G be a graph with n vertices. There exists a natural number $k_0 \leq n$ such that $\chi_{k_0}(G) \leq \chi_k(G)$ for all $k = 1, 2, \ldots$.

Proof. Let F denote the family of all functions on the vertex set of G that takes real values between 0 and 1. For $f \in F$ define $M(f) = \min_{u \operatorname{adj} v}|f(u) - f(v)|_1 \leq \frac{1}{2}$. Let $X = 1/\sup_{f \in F} M(f)$. Then $\chi_k(G) = k/\max_{c \in C_k} \min_{u \operatorname{adj} v}|c(u) - c(v)|_k = 1/\max_{c \in C_k} \min_{u \operatorname{adj} v}(1/k)\,|c(u) - c(v)|_k = 1/\max_{c \in C_k} \min_{u \operatorname{adj} v}|(c(u)/k) - (c(v)/k)|_1 \geq X$. The last equality results from scaling, i.e., $(1/k)\ |a|_k = |a/k|_1$ for all $a \in Z_k$. It is now sufficient to show that $X \geq \chi_k(G)$ for some $k \leq n$. This is done in several steps.

We first claim that there exists a function $\phi \in F$ such that $X = 1/M(\phi)$, i.e., that the supremum of M is achieved. Since the values of f are modulo 1, f can be regarded as an n-tuples of real numbers and $M(f)$: $S^1 \times S^1 \times \cdots \times S^1 \rightarrow R$ as a function from the product of n copies of the 1-sphere to the reals.

This function, defined on a compact domain, is clearly continuous, and therefore achieves a maximum.

For a real function f: $V \to R$, let $H(f)$ be a directed graph on the same vertex set as G and defined as follows: There is an arc (u, v) directed from u to v if

(1) $M(f) = |f(u) - f(v)|_1$

and

(2) There is an $\varepsilon > 0$ such that $M(h) < M(f)$ for any function h such that $h(x) = f(x)$ for all $x = u$ and $f(u) < h(u) < f(u) + \varepsilon$.

The first condition says that the minimum is achieved between vertices u and v. The second condition says that slightly increasing the value of f at u decreases $M(f)$.

It is next shown that for any function f there is a function g such that $H(g)$ is a connected graph and $M(f) = M(g)$. Proceed by induction. Assume that H is disconnected and H_0 is a connected component of H with vertex set V_0. Then for any adjacent $u \in V_0$ and $v \in V - V_0$ it holds that $|f(u) - f(v)|_1 > M(f)$. Uniformly decreasing all values of $f(x)$, $x \in V_0$, until the first occurrence of $|f(u) - f(v)|_1 = M(f)$ for some adjacent $u \in V_0$ and $v \in V - V_0$, introduces an arc joining vertex u to v. This decreases the number of components of H. In the remainder of the proof H will be assumed connected.

The graph $H(\phi)$ contains a directed cycle. By way of contradiction, assume $H(\phi)$ contains no directed cycle. Then there exists a vertex v_1 with outdegree $(v_1) = 0$. Removing v_1 and repeating this argument gives an ordering of the vertices $(v_1, \ldots, v_n)$ such that, for all i, outdegree $(v_i) = 0$ in the subgraph induced by vertices $\{v_i, \ldots, v_n\}$. Define f: $V \to R$ by $f(v_i) = \phi(v_i) + \varepsilon/i$. Then $|f(u) - f(v)|_1 > |\phi(u) - \phi(v)|_1$ for vertices u and v adjacent in $H(\phi)$, and for $\varepsilon > 0$ sufficiently small, $|f(u) - f(v)|_1 > M(\phi)$ for nonadjacent pairs of vertices. Then for sufficiently small $\varepsilon > 0$, $M(f) > M(\phi)$, contradicting the maximality of ϕ.

Each arc (u, v) in $H(\phi)$ implies that

$$\phi(v) - \phi(u) = \begin{cases} M(\phi) & \text{or} \\ M(\phi) - 1. \end{cases} \tag{1}$$

Let $v_1, \ldots, v_{k_0}$ be a directed cycle in $H(\phi)$. We will show that $X \geq \chi_{k_0}$. Apply (1) consecutively to each arc of this cycle and sum to obtain that $k_0 M(\phi) = d$, where d is an integer. Let u_0 be some fixed vertex and assume, without loss of generality, that $\phi(u_0) = 0$. If $\phi(u_0) \neq 0$ then $\phi(u_0)$ can be subtracted (modulo 1) from all values of ϕ. Let $u_0, u_1, \ldots, u_p$ be a path (not necessarily directed) from vertex u_0 to an arbitrary vertex $u = u_p$. Then for $0 \leq i < p$, $\phi(u_{i+1}) - \phi(u_i) = \pm M(\phi)$ or $\pm(M(\phi) - 1)$. Summing this formula successively for each vertex $u_0, u_1, \ldots, u_{p-1}$ along the path yields $\phi(u_p) = aM(\phi) + b = (ad/k_0) + b$, where a, b, and d are integers. In other words, $\phi(u_p) \equiv f(u_p)/k \pmod 1$ for some integer $f(u_p)$ depending on u_p. Finally, $X = 1/\min|\phi(u_p) - \phi(v)|_1 = 1/\min|(f(u_p)/k_0) - (f(v)/k_0)|_1 = k/\min|f(u_p) - f(v)|_{k_0} \geq \chi_{k_0}(G)$. ∎

3. STAR CHROMATIC NUMBER

In light of Theorem 2, define the *star chromatic number* $\chi^\star(G)$ as the least of the Z_k-chromatic numbers. If G is a connected graph with n vertices,

$$\chi^\star(G) = \min_{1 \leq k \leq n} \chi_k(G).$$

In a sense, $\chi^\star(G)$ corresponds to the best possible coloring of G, which may be better than the coloring corresponding to the ordinary chromatic number. In fact, Theorem 3 shows more—that $\chi^\star$ cannot even be improved if the colorings are allowed to take on any real values (using the circular norm between colors). In Examples 1 and 2 of Section 2, the star chromatic numbers are $2\frac{1}{2}$ and $3\frac{1}{2}$, respectively, whereas the ordinary chromatic numbers are 3 and 4. The next result states that the star chromatic number cannot be too far from the ordinary chromatic number. When the graph is understood, we abbreviate $\chi = \chi(G)$, $\chi_k = \chi_k(G)$, and $\chi^\star = \chi^\star(G)$.

Theorem 4. For all graphs $\chi - 1 < \chi^\star \leq \chi$.

Proof. The second inequality follows from Theorem 1. If $k = \chi$, then $\chi^\star \leq \chi_k = \chi$. For the first inequality assume, by way of contradiction, that $k_0/\min|c(u) - c(v)|_{k_0} = \chi^\star \leq \chi - 1$ for some Z_{k_0}-coloring c. This implies that $|c(u) - c(v)|_{k_0} \geq k_0/(\chi - 1)$ for all adjacent vertices u and v. Now define a related coloring c' by $c'(u) = \lceil c(u)(\chi - 1)/k_0 \rceil$. We show that c' is a proper $\chi - 1$ coloring of G, contradicting the fact that $\chi(G)$ is the chromatic number of G. Since $1 \leq c(u) \leq k_0$ then $1 \leq c'(u) \leq \chi - 1$. Also, if u and v are adjacent, then $|c'(u) - c'(v)| = |\lceil c(u)(\chi - 1)/k_0 \rceil - \lceil c(v)(\chi - 1)/k_0 \rceil| > |(c(u) - c(v))(\chi - 1)/k_0| - 1 \geq 0$. The first inequality follows from the fact that $|\lceil a \rceil - \lceil b \rceil| > |a - b| - 1$ for all real a and b. ∎

Lemma 1. If H is a subgraph of G then $\chi_k(G) \geq \chi_k(H)$ for all k, and $\chi^\star(G) \geq \chi^\star(H)$.

Proof. This is immediate from the fact that any Z_k-coloring $c: V(G) \to Z_k$ can be restricted to such a coloring on $V(H)$. ∎

Theorem 5. Let G be a connected graph with at least two vertices. Then $\chi^\star(G) \geq 2$ with equality if and only if G is bipartite.

Proof. If follows directly from the definitions that $\chi^\star(G) \geq 2$. If G is bipartite then, by Theorem 4, $\chi^\star(G) \leq \chi(G) = 2$. Therefore $\chi^\star(G) = 2$. Conversely, if $\chi^\star(G) = 2$, then again by Theorem 4, $2 = \chi^\star(G) > \chi(G) - 1$. Therefore $\chi(G) < 3$, which means that $\chi(G) = 2$. Hence G is bipartite. ∎

Define a graph $G_{m,n}$, $2 \leq 2m < n$, as follows: The vertex set is $V = \{1, 2, \ldots, n\}$. Vertex i is adjacent to vertex j if and only if $|i - j|_n \geq m$. Note that $G_{1,n} = K_n$, the complete graph, and $G_{m,2m+1} = C_{2m+1}$, the odd cycle. Theorem 6 implies that the range of $\chi^\star$ is the set of all rational numbers satisfying the necessary condition $\chi^\star \geq 2$ of Theorem 5. The proof requires the following lemma: Let $X = \{1, 2, \ldots, n\}$ be a set of points uniformly distributed on a circle S^1 of circumference n such that consecutive points are 1 unit apart. For a positive integer $m < n/2$, call a function $f: X \to S^1$ m-*expanding* for Z_n if $|f(i) - f(j)|_n > m$ whenever $|i - j|_n \geq m$. (Note that $|f(i) - f(j)|_n = m$ is not allowed here.)

Lemma 2. Let m and n be integers such that $1 \leq m < n/2$. Then there cannot exist an m-expanding function for Z_n.

Proof. Let $n = mq + r$, $0 \leq r < m$, and $m = rq' + s$, $0 \leq s < r$. First consider the case $r = 0$. Assume, by way of contradiction, that an m-expanding function exists. Then all pairs of points among $\{m, 2m, \ldots, qm\}$ are at a distance greater than or equal to m. Hence all pairs of points among $\{f(m), f(2m), \ldots, f(qm)\}$ are at a distance greater than m. But this is impossible.

Now assume $r > 0$. We proceed by induction on m. We just proved that the theorem is true if m is a divisor of n and, in particular, if $m = 1$. It will be shown that if f is m-expanding, then f is s-expanding. If $s = 0$, then this is an immediate contradiction. If $s > 0$, then $1 \leq s < m$ and the lemma would follow by induction. A function f is called r-*contracting* if $|f(i) - f(j)|_n < r$ whenever $|i - j|_n \leq r$. It will be shown that if f is m-expanding then f is r-contracting, and if f is m-expanding and r-contracting, then f is s-expanding. Let i and j be any pair of points in X with $|i - j|_n \leq r$. Consider the set $A = \{i, i + m, \ldots, i + (q - 1)m\}$ and let the closest points to $f(i)$ in $f(A)$ (one on each side) be $f(u)$ and $f(v)$. Since all pairs of points in A are at a distance greater than or equal to m, and since f is m-expanding, $|f(i) - f(u)|_n > m$ and $|f(j) - f(v)|_n > m$. Consider the set $B = \{j, i + m, \ldots, i + (q - 1)m\}$, which is exactly the set A with the single point i replaced by j. The same argument as above shows that $|f(j) - f(k)|_n > m$ for all $k \in B - \{j\}$. Points $f(u)$ and $f(v)$ divide S^1 into two arcs. We claim that $f(i)$ and $f(j)$ lie on the same arc. If not, the points $f(A) \cup \{f(j)\}$ divide S^1 into $q + 1$ arcs, each of length greater than m. Thus $n \geq (q + 1)m > qm + r = n$, a contradiction. Without loss of generality it may now be assumed that $f(u) < f(i) \leq f(j) < f(v)$. The assumption that f is m-expanding implies that $f(v) - f(i) < 2m + r$. Therefore $f(i) - f(j) < 2m + r[(f(i) - f(u)) + (f(v) - f(j))] < 2m + r - m - m = r$. and f is r-contracting.

To show that f is s-expanding, let i and j be any pair of points in X with $i - j|_n \geq s$. If $|i - j|_n \geq m$, then $|f(i) - f(j)|_n > m > s$, because f is m-expanding. So assume, without loss of generality, that $s \leq i - j < m$. Let a be the point of X such that $|i-j|_n + |j-a|_n = |i-a|_n = m$. Consider the least integer α such that $a + \alpha r < j \leq a + (\alpha + 1)r$. Since $|j-a|_n \leq m - s = rq'$, note that $\alpha + 1 \leq q'$. Because f is m-expanding, $|f(i) - f(a)|_n > m$. Because f is r-contracting, $|f(j) - f(a)|_n < (\alpha + 1)r$. Therefore $|f(i) - f(j)|_n > m - (\alpha + 1)r = (q'r + s) - (\alpha + 1)r = (q' - \alpha - 1)r + s \geq s$, so that f is s-expanding. Since $s < m$, this contradicts the induction hypothesis and f is, therefore, not f-expanding. ∎

Theorem 6 For $1 \leq m < n/2$ we have $\chi^\star(G_{m,n}) = n/m$.

Proof. We must show that $\min_{1\leq k\leq n} \min_f \psi(f) = \min_{1\leq k\leq n} \chi_k(G) = \chi^\star(G) = n/m$. For this it is sufficient to show that $\psi(f) = s/\min_{i\,\mathrm{adj}\,j}|f(i) - f(j)|_s \geq n/m$ for any Z_s-coloring f of $G_{m,n}$. This is equivalent to showing $m \geq \min|nf(i)/s - nf(j)/s|_n$. Define $g(i)) = nf(i)/s$, and by way of contradiction, assume that $|g(i) - g(j)|_n > m$ for all adjacent pairs of vertices i and j. But i and j are adjacent in $G_{m,n}$ if and only if $|i-j| \geq m$. The result now follows from Lemma 2. ∎

The following result is a direct consequence of Theorem 6 and the fact that $K_n = G_{1,n}$ and $C_{2n+1} = G_{n,2n+1}$.

Corollary 1. For the complete graphs and odd cycles

(a) $\chi^\star(K_n) = n$
(b) $\chi^\star(C_{2n+1}) = 2 + (1/n)$.

A clique is a complete subgraph of G. The clique number $\omega(G)$ is the maximum order of a clique in G. It is obvious that $\chi(G) \geq \omega(G)$. In the case $\chi(G) = \omega(G)$, the star chromatic and the ordinary chromatic numbers coincide.

Theorem 7. If $\chi(G) = \omega(G)$, then $\chi^\star(G) = \chi(G)$.

Proof. Assume that $\chi(G) = \omega(G)$. Use Theorems 4, Lemma 1, and Corollary 1, respectively: $\omega(G) = \chi(G) \geq \chi^\star(G) \geq \chi^\star(K_{\omega(G)}) = \omega(G)$. ∎

Remark. The converse does not hold. There are graphs for which $\chi^\star = \chi$, but $\chi^\star \neq \omega$. One example is the Grötzsch graph, the smallest 4-chromatic graph with no triangles, and another is the 3-chromatic Petersen graph. In both cases the star chromatic number is obtained from the usual 4 and 3-colorings, respectively.

Theorem 4 bounds the star chromatic number in terms of the ordinary chromatic number. Theorem 8 gives bounds on the star chromatic number in terms of the Z_k-chromatic numbers.

Lemma 3. For any natural numbers j, k we have $1/\chi_j - 1/k < 1/\chi_k < 1/\chi_j + 1/j$.

Proof. By definition $1/\chi_k = (1/k)\max_c \min |c(u) - c(v)|_k$. Let c be a Z_k-coloring that achieves the maximum. Define a Z_j-coloring c' by $c'(u) = \lfloor c(u)j/k \rfloor$. It is sufficient to prove that $(1/k)|c(u) - c(v)|_k < (1/j)|c'(u) - c'(v)|_j + 1/j$ for all adjacent vertices u and v. Now $(1/k)|c(u) - c(v)|_k = (1/j)|c(u)j/k - c(v)j/k|_j < (1/j)||\lfloor c(u)j/k\rfloor - \lfloor c(v)j/k\rfloor||_j + (1/j) = (1/j)|c'(u) - c'(v)|_j + 1/j$. The first equality is just a change of scale. The second inequality follows from the following fact about ordinary absolute value: $|a - b| - ||\lfloor a\rfloor - \lfloor b\rfloor|| < 1$ for all real a and b. The lower bound is obtained by reversing the role of j and k. ▮

Theorem 8. For any natural number k we have

$$\chi_k \geq \chi^\star > 1\Big/\left(\frac{1}{\chi_k} + \frac{1}{k}\right).$$

Proof. The upper bound is obvious from the definition of $\chi^\star$. The lower bound follows from Lemma 3. ▮

Corollary 2. $\lim_{k\to\infty} \chi_k = \chi^\star$.

Proof. The limit of both the upper and lower bound on $\chi^\star$ in Theorem 8 is $\lim_{k\to\infty} \chi_k$. ▮

4. OPEN QUESTIONS

Sections 2 and 3 discuss only basic properties of the star chromatic number. Many problems remain open, for example:

1. What determines whether $\chi^\star = \chi$?
2. Besides the odd cycles, what are the planar graphs G with $2 < \chi^\star(G) < 3$? By results of the previous section, it is necessary that G be 3-chromatic and contain no triangles.
3. What are some infinite families of planar graphs with $3 < \chi^\star < 4$? Do all edge critical 4-chromatic graphs fall into this category?
4. The k-chromatic numbers have algebraic structure not possessed by the ordinary chromatic number, namely that of Z_k. In general, can the star chromatic number be applied to problems concerning the ordinary chromatic number?

ACKNOWLEDGMENT

The author would like to thank the referees for their helpful comments.

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