# The Classification of Closed Surfaces Using Colored Graphs 

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#### Abstract

A short graph theoretic proof of the classification of closed surfaces is given. The new proof has the feature that the symmetric canonical graphs for the surfaces do not correspond to the canonical polygons in the usual proof of the classification.


## 1. Introduction

Surface, in this article, will always mean a compact, connected 2-manifold without boundary. Surfaces are classified according to orientation, i.e., whether or not there is a coherent clockwise or counterclockwise orientation on the surface. For example, the sphere $S$ and the torus $T$ are orientable. The projective plane $P$, which is defined as a disk with antipodal boundary points identified, is nonorientable. All other surfaces can be defined from $S, T$ and $P$ using only the connected sum operation. The connected sum $F_{1} \# F_{2}$ is defined by removing a disk from each of the surfaces $F_{1}$ and $F_{2}$ and identifying the two circles that are the boundaries of those disks. For example $T_{2}=T \# T$ is the genus-two torus and $K=P \# P$ is the Klein bottle. Let $m F$ denote the connected sum of $m$ copies of the surface $F$. Then the list of orientable surfaces is: $S$ and $g T, g=1,2, \ldots$, and the list of nonorientable surfaces is: $g P, g=1$, $2, \ldots$ That these lists are complete was originally proved by Dehn and Heegaard (1907), and a classic exposition of the proof, followed in several newer texts, is due to Seifert and Threlfall [3]. Notice that $T \# P$ is not on this list. That is because $T \# P \approx 3 P$, a fact that is not completely obvious and will be mentioned again in Sect. 3. Throughout $\approx$ will denote homeomorphism of surfaces.

The purpose of this article is to give a simple, graph theoretic proof of the classification of surfaces. In addition to giving a short proof, the method yields new canonical forms for surfaces. Surfaces will be encoded as graphs (Sect. 2) and all manipulation of the surface will be done in terms of just one basic graph operation - fusion (Sect. 3). Propositions 1-4 in Sects. 2 and 3 provide the link between results about graphs and results about surfaces. Given this translation, the short graph theoretic Sects. 4 and 5 contain the core results about invariants, canonical forms and the classification of surfaces. The graph theoretic approach to manifolds has developed over the past several years and references [2] and [4] provide surveys and extended lists of references.

## 2. Surface and 3-Graphs

The main objects of investigation in this paper are 3-graphs. A 3-graph is a graph, regular of degree 3 , with a proper edge coloring in 3 colors. A 3-graph need not be connected and may have multiple edges. To every 3 -graph $G$ is associated a surface $\Delta G$ obtained by attaching a 2 -cell to each 2 -colored cycle of $G$. For the first graph $G$ in Fig. 1 the surface $\Delta G$ is the sphere. A little mental pasting will suffice to convince one that the surfaces encoded by the other 3-graphs in Fig. 1 are the projective plane and torus, respectively. The graphs in Fig. 1 will be denoted $S, P$ and $T$ to conform to the notation for the corresponding surfaces. If a color is understood, then often the label in the figure will be omitted. The first proposition states that every surface can be encoded by a 3-graph. In this paper it is assumed that a surface can be triangulated, a fact that itself is not trivial.


Fig. 1. Encodings of the sphere, projective plane and torus

Proposition 1. For every surface $F$ there is a 3-graph $G$ such that $F \approx \Delta G$.
Proof. Let $\Delta$ be a triangulation of $F$. Consider the barycentric subdivision $\Delta^{\prime}$ of this triangulation, where each vertex in $\Delta^{\prime}$ is labeled 0,1 or 2 according to the dimension of the simplex in $\Delta$ that $v$ represents. Then each triangle in $\Delta^{\prime}$ gets all three labels $0,1,2$. The dual graph of $\Delta^{\prime}$ with the edges appropriately labeled in $\{0,1,2\}$ is the desired 3-graph $G$.

Our perspective now is that a 3-colored graph "is" a surface. A graph is called bipartite if the vertex set can be partitioned into two parts $A$ and $B$ such that all edges join vertices of $A$ to vertices of $B$. For a 3-graph $G$ let $\Theta(G)$ take values in the set \{bipartite, non-biparte\}.

Proposition 2. $\Theta(G)=$ bipartite if and only if $\Delta G$ is an orientable surface.
Proof. The dual graph of $G$ on the surface yields a triangulation of $\Delta G$. Each edge in this triangulation can be colored 0,1 or 2 corresponding to the edge in $G$ that it crosses. Each triangle in the triangulation then has sides colored 0,1 and 2. Now the triangles corresponding to the vertices in the first part $A$ are oriented $(0,1,2)$ while those in the second part $B$ are oriented ( $0,2,1$ ).

## 3. Fusion

The basic operation on 3-graphs is fusion. A 3-graph $G^{\prime}$ is obtained from $G$ by fusion on vertices $u$ and $v$ if
(a) Vertices $u$ and $v$ (and all edges joining them) are removed from $G$
(b) The "free" ends of edges of the same color are then identified to form $G^{\prime}$

Fusion, introduced by Ferri and Gagliardi [1], may occur in various ways. Two cases are considered below. If $J$ is a subset of colors, let $G_{J}$ denote the subgraph of $G$ induced by edges with colors in $J$, and $G_{\bar{J}}$ the complementary subgraph induced by edges with colors not in $J$. For example if $|J|=1$, then $G_{J}$ is a 1 -factor of $G$ and $G_{\bar{J}}$ is the disjoint union of 2-colored cycles, each called a bigon.

Case 1 . Vertices $u$ and $v$ are adjacent, joined by (one or more) edges with color set, say $J$. Furthermore $u$ and $v$ lie in distinct components of the complementary subgraph $G_{j}$. If $G^{\prime}$ is obtained from $G$ by fusion on $u$ and $v$ then $G$ and $G^{\prime}$ are called elementary equivalent. The vertices $u$ and $v$ together with the edges joining them is called a dipole, and we say that $G^{\prime}$ is obtained from $G$ by removing a dipole or that $G$ is obtained from $G^{\prime}$ by adding a dipole. Furthermore, graphs $G$ and $G^{\prime}$ are called equivalent if there is a sequence $G=G_{0}, G_{1}, \ldots, G_{n}=G^{\prime}$ such that $G_{i-1}$ and $G_{i}$ are elementary equivalent for $i=1,2, \ldots, n$. Equivalence of 3 -graphs will be denoted $G \approx G^{\prime}$. An example of equivalent 3 -graphs is shown in Fig. 2 where dipole $\{i, j\}$ is added and then dipole $\{a, e\}$ is removed. Removing a (single edge) dipole in $G$ corresponds in $\Delta G$ to merging two distinct faces through an edge joining them, and adding a dipole corresponds to identifying an interior part of two edges of different colors on the boundary of the same face. This makes the following proposition clear in one direction, and explains the coincidence of the notation for equivalence of 3 -graphs and homeomorphism of surfaces. The proof in the other direction, which is not needed in our proof of the classification, appears in [1].


Fig. 2. Equivalent 3-graphs

Proposition 3. $\Delta G \approx \Delta G^{\prime}$ if and only if $G \approx G^{\prime}$.
Case 2. The graph $G$ is disconnected with components $G_{1}$ and $G_{2}$, and $u$ and $v$ are in different components. If $G^{\prime}$ is obtained from $G$ by fusion on vertices $u$ and $v$ then $G^{\prime}$ is denoted $G_{1} \stackrel{u v}{\#} G_{2}$ and is called the connected sum of $G_{1}$ and $G_{2}$. Examples are
shown in Fig. 3. Connected sum has the following meaning with respect to the associated surfaces. Remove small 2 -simplices centered at $u$ and $v$ in $\Delta G_{1}$ and $\Delta G_{2}$, respectively; then indentify the boundary triangles of these two 2 -simplicies so that like colored free edges of $G$ are associated. Therefore we have

Proposition 4. $\Delta\left(G_{1} \# G_{2}\right) \approx \Delta G_{1} \# \Delta G_{2}$.


T


P


T\#P
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Fig. 3. Connected sum of 3-graphs

It has tacitly been assumed in Proposition 4 that the connected sum of 3-graphs does not depend on the vertices $u$ and $v$. This is true, but not obvious. Even for surfaces there are different ways to attach the boundary circles of disks, and it is not at all obvious that the two resulting connected sums are homeomorphic. For this paper it is sufficient to prove the independence of the choice of $u$ and $v$ for the special cases in Lemma 1a. However the general case follows from Corollary 2 at the end of this paper. In analogy to surfaces, the notation $m G$, will denote the connected sum of $m$ copies of the 3-graph $G$. Recall that $S, P$ and $T$ are the graphs in Fig. 1. By convention let $0 T=S$.

## Lemma 1.

(a) For any 3-graph $G$ the connected sums $G \# P$ and $G \# T$ are independent of the vertices of fusion.
(b) If $G \approx G^{\prime}$ then $G \# P \approx G^{\prime} \# P$ and $G \# T \approx G^{\prime} \# T$.
(c) $T \# P \approx 3 P$.

Proof. Part (a) will be proved for $G \# P$ only, because the proof for $G \# T$ is analogous. Because of the symmetry of $P$, it is sufficient to show that if $v$ and $v^{\prime}$ are two adjacent vertices of $G$ and $u$ is any vertex of $P$, then $G \stackrel{u v}{\#} P \approx G \stackrel{u v^{\prime}}{\#} P$. This

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Fig. 4. Part (a) of Lemma 1


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equivalence is shown in Fig. 4, where the two elementary equivalences are, by first adding dipole $\{a, b\}$ and then removing dipole $\{c, d\}$. It is sufficient to prove part (b) in the case that $G$ and $G^{\prime}$ are elementary equivalent. Since the fusion vertices can be shown arbitrarily by part (a), they can be chosen to avoid the dipole that is added or removed. Concerning part (c) Fig. 2 and 3 together constitute a proof that $T \# P \approx 3 P$.

Propositions 1-4 provide links between the topology of surfaces and the combinatorics of 3 -graphs. For example part (c) of Lemma 1 translates precisely to the result mentioned in the introduction. Subsequent results and proofs will be completely graph theoretic and will not rely on these propositions.

## 4. Invariants and Canonical Forms

The two important invariants of 3-graphs are $\Theta$ and $\chi$. The number of vertices in a 3-graph is even, so let $n=n(G)$ denote half the number of vertices in $G$. Let $r=r(G)$ be the total number of bigons in $G$, that is, the number of 2 -colored cycles with colors 01,02 or 12 . Now define $\chi(G)=r-n$. The notation $\chi$ is to suggest that this invariant is an analogue of the Euler characteristic of a surface. It is straightforward to check that indeed both $\Theta$ and $\chi$ are invariants by showing that each is preserved under elementary equivalence.

Lemma 2. If $G_{1} \approx G_{2}$ then $\Theta\left(G_{1}\right)=\Theta\left(G_{2}\right)$ and $\chi\left(G_{1}\right)=\chi\left(G_{2}\right)$.
If no dipoles can be removed from a 3-graph $G$, then $G$ is said to be reduced. Every 3-graph is equivalent to a reduced 3-graph by merely removing dipoles until no longer possible. Note that in a reduced 3-graph $r(G)=3$ and therefore $\chi(G)=$ $3-n$. Two special families of reduced graphs, $T_{n}, n \geq 1, n$ odd, and $U_{n}, n \geq 2$, are referred to as canonical 3-graphs. Both are formed from a single bigon with $2 n$ vertices. In $T_{n}$, chords of the third color join each vertex to the diametrically opposite vertex. In $U_{n}$ there is one diagonal chord with the remaining chords perpendicular to it. Graphs $T_{5}$ and $U_{5}$ are shown in Fig. 5. Note that $\Theta\left(T_{n}\right)=$ bipartite and $\Theta\left(U_{n}\right)=$ non-bipartite. Also $\chi\left(T_{n}\right)=\chi\left(U_{n}\right)=3-n$. Note also that $T_{1}=S ; U_{2}=P$; and $T_{3}=T$.


Fig. 5. Canonical 3-graphs

Lemma 3. $T_{n} \approx\left(\frac{n-1}{2}\right) T$ for $n=1,3, \ldots$, and $U_{n} \approx(n-1)$ P for $n=2,3, \ldots$.
Proof. Proceeding by induction, it is sufficient to show that $T_{n-2} \# T \approx T_{n}$ and $U_{n-1} \# P \approx U_{n}$. The first equivalence is shown in Fig. 6, where dipole $\{c, d\}$ is added and dipole $\{a, b\}$ is removed. The second equivalence is almost immediate.


Fig. 6. $T_{n-2} \# T=T_{n}$

Lemma 3 and Proposition 4 imply that the canonical 3-graphs encode all the surfaces described in the introduction.

## 5. The Classification of 3-Graphs.

The main results, Theorem 1 and its corollaries, classify 3-graphs up to equivalence. In view of Propositions 1-4, the classification of surfaces is a direct consequence.

Theorem 1. Every 3-graph is equivalent to a unique $n T, n=0,1, \ldots$, or $n P, n=1$, $2, \ldots$.

Proof. Let $G$ be a 3-graph and proceed by induction on $n=n(G)$. There is no loss of generality in assuming that $G$ is reduced. A reduced 3-graph consists of a bigon with chords. Two cases are now considered separately: $\Theta(G)=$ bipartite and $\Theta(G)=$ nonbipartite.

In the bipartite case we prove that $G \approx\left(\frac{n-1}{2}\right) T$. If $n=1$ there is only one possibility $G \approx S=0 T$. In general refer to Fig. 7 for the bipartite case. Straight lines in the figure signify edges in the graph and curves signify paths. Two of the chords must cross each other if $G$ is to have only 3 bigons. Adding dipole $\{h, i\}$, removing dipole $\{a, e\}$, and adding dipole $\{j, k\}$ results in $G \approx G^{\prime} \# T$. But $G^{\prime}$ remains reduced, bipartite and $n\left(G^{\prime}\right)=n-2$, so, by induction, $G^{\prime} \approx\left(\frac{n-3}{2}\right) T$.

In the non-bipartite case we prove that $G \approx(n-1) P$. If $n=2$ there is only one possibility $G=P$. In general, refer to Fig. 8 for the nonbipartite case. Because $G$ is nonbipartite there must be a chord with colors 0 and 1 on its left, and because there are an odd number of vertices to the right of this chord there must exist a chord

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Fig. 7. Bipartite case
$\{c, d\}$ that crosses $\{a, b\}$. After adding dipole $\{e, f\}$ and removing dipole $\{c, d\}$, $G \approx G^{\prime} \# P$ where $G^{\prime}$ is reduced and $n\left(G^{\prime}\right)=n-1$. By induction $G^{\prime} \approx(n-2) P$ or $G^{\prime} \approx\left(\frac{n-2}{2}\right) T \# P \approx(n-1) P$. The last equivalence is proved by repeated application of $T \# P \approx 3 P$, which is part (c) of Lemma 1. Uniqueness in Theorem 1 folows from Lemma 3 and the fact that no two distinct canonical 3-graphs are equivalent.

Corollary 1. Every 3-graph is equivalent to a unique canonical 3-graph.
Proof. Corollary 1 follows immediately from Lemma 3 and Theorem 1.
Corollary 2. A 3-graph is uniquely determined up to equivalence by the invariants $\Theta$ and $\chi$.

Proof. Assume that $G_{1}$ and $G_{2}$ have the same invariants and that $G_{1}^{\prime}$ and $G_{2}^{\prime}$ are the canonical graphs that are, by Corollary 1, equivalent to $G_{1}$ and $G_{2}$, respectively.


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Fig. 8. No-bipartite case


Since canonical graphs are uniquely determined by $\Theta$ and $\chi$ we have $G_{1} \approx G_{1}^{\prime} \approx$ $G_{2}^{\prime} \approx G_{2}$.

Corollary 3. A reduced 3-graph is uniquely determined by $\Theta$ and $n$.
Proof. Corollary 3 follows from Corollary 2 because, for a reduced 3-graph, $\chi=$ 3-n.

Corollaries 2 and 3 translate, in light of Proposition 3, to the fact that a surface is uniquely determined up to homeomorphism by orientability and the Euler characteristic.

## References

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