

Combinatorial Maps

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The classical approach to maps, as surveyed by Coxeter and Moser ("*Generators and Relations for Discrete Groups*," Springer-Verlag, 1980), is by cell decomposition of a surface. A more recent approach, by way of graph embedding schemes, is taken by Edmonds (*Notices Amer. Math. Soc.* **7** (1960), 646), Tutte (*Canad. J. Math.* **31** (5) (1979), 986–1004), and others. Our intention is to formulate a purely combinatorial generalization of a map, called a combinatorial map. Besides maps on orientable and nonorientable surfaces, combinatorial maps include polytopes, tessellations, the hypermaps of Walsh, higher dimensional analogues of maps, and certain toroidal complexes of Coxeter and Shephard (*J. Combin. Theory Ser. B.* **22** (1977), 131–138) and Grünbaum (Colloques internationaux C.N.R.S. No. 260, Problèmes Combinatoire et Théorie des Graphes," Orsay, 1976). The concept of a combinatorial map is formulated graph theoretically. The present paper treats the incidence structure, the diagram, reducibility, order, geometric realizations, and group theoretic and topological properties of combinatorial maps. Another paper investigates highly symmetric combinatorial maps.

1. INTRODUCTION

A *polytope* is the convex hull of a finite set of points in Euclidean space E^n . A supporting hyperplane of a polytope P is a hyperplane that intersects P in such a way that P lies in one of the closed half spaces determined by the hyperplane. Intersections of a polytope P with supporting hyperplanes are polytopes. With the exception of P itself, these are called faces. The faces of dimension k are k -faces. The set of all faces of an n -dimensional polytope P forms a cell complex of dimension $n - 1$, called the *boundary complex of P* . The polytopes of dimension 2 and 3 are polygons and polyhedra, respectively.

A map on a surface, i.e., a cell decomposition of a surface is a topological generalization of the boundary complex of a polyhedron. Several authors have further extended the concept of a map. Such extensions include permutation maps [1, 5, 9, 15, 18], the "combinatorial polytopes" of McMullen [12], toroidal complexes of Coxeter and Shephard [4] and the

“polystromas” of Grünbaum [8]. In this paper a generalization of a map on a surface, called a combinatorial map, will be formulated in terms of edge colored graphs. This generalization originally appeared in the author’s work [19]. At the time of writing, the relationship to the work of J. Tits on incidence structures, chamber complexes and chamber systems became known. Some of this is treated in Section 3. We acknowledge the influence of Tits’s research [16, 17]. Another related concept, called a crystallization, was independently investigated in a topological setting by Ferri [6] and Gagliardi [7]. It was rediscovered by Lins [11], where it is called a graph-encoded map.

Associated with each combinatorial map G is an incidence structure $S(G)$, a diagram $D(G)$ and a topological space $|G|$. Definitions and examples are given in Section 2. The diagram $D(G)$ of a combinatorial map generalizes the classical Schläfli symbol of a polytope. Irreducibility of a combinatorial map is characterized by a connected diagram. The faces of a polytope are partially ordered by inclusion. A class of combinatorial maps, called ordered maps, includes the polytopes and most other classical examples, and it is shown that these are characterized by a linear diagram. The incidence structure $S(G)$ generalizes the facial structure of a polytope.

The examples that motivated this paper are the maps on surfaces. There is a vast literature on this subject, [1, 3, 10, 15, 18] being a sample. An agreeable outcome of the theory of combinatorial maps is that the cell decomposition and embedding schemes approaches to maps on surfaces become unified. In Section 5 it is shown that the ordered rank 3 combinatorial maps are exactly the maps on surfaces. The set of all rank 3 combinatorial maps corresponds to the hypermaps of Walsh [21]. More generally, a cell decomposition of a manifold yields an ordered combinatorial map, but for rank >3 not every ordered combinatorial map can be so realized.

Every combinatorial map G has an underlying topological space $|G|$. Topological properties of $|G|$, such as orientability, the fundamental group and coverings are related to the combinatorial properties of G in Section 6. Ramified coverings have been important in the theory of maps on surfaces, in the proof of the Heawood map coloring theorem, and more recently in the theory of chamber complexes and systems [13, 17]. The latter ideas of Tits and Ronan are incorporated in order to develop a combinatorial analogue of topological ramified covering space theory. A sequence $\pi^1(G) \rightarrow \pi^2(G) \rightarrow \dots \rightarrow \pi^{n-1}(G) \cong \pi_1(|G|)$ of combinatorial fundamental groups is defined such that each is a refinement of its successor. The coverings of a given combinatorial map are shown to be in one-to-one correspondence with permutation representations of the appropriate combinatorial fundamental group. This result is used in [20] to construct highly symmetric combinatorial maps.

In Section 7 an equivalent formulation of a combinatorial map, the Schreier representation, is given in terms of a group generated by involutions. With respect to this formulation, expressions are obtained for the automorphism group, the fundamental groups and for universal covers of a combinatorial map.

2. COMBINATORIAL MAPS

To motivate the definition of a combinatorial map consider the boundary complex $B(P)$ of a 3-dimensional polytope P . The definition of a combinatorial map will reflect the following essential property of $B(P)$:

(*) Every edge is incident with exactly two vertices; on a given face each vertex is incident with exactly two edges; every edge is incident with exactly two faces.

Let ΔP denote the barycentric subdivision of $B(P)$. The three vertices of any 2-simplex in ΔP can be labeled 0, 1, and 2 according to whether the vertex represents a 0, 1, or 2-face of P . Now form a labeled graph $G(P)$ as follows: The points of $G(P)$ are the 2-simplexes of ΔP and two distinct points are joined by a line labeled i if and only if the respective 2-simplexes have a common edge without label i . The graph $G(P)$ completely determines ΔP because ΔP can be retrieved by “glueing” together labeled 2-simplexes that correspond to adjacent points of the graph and making the appropriate identifications. Property (*) of the polyhedron P has an equivalent interpretation in terms of the graph $G(P)$:

(**) Every point of $G(P)$ is incident with exactly one line labeled i for $i = 0, 1, 2$.

Turning to the general situation, let I be a finite set. A *combinatorial map over I* is a connected graph G , regular of degree $|I|$, whose lines are $|I|$ -colored such that no two incident lines are the same color. A combinatorial map may be finite or infinite. Let the function $\tau: E(G) \rightarrow I$, from the line set of G to I , be the coloring. The image of a line or set of lines under τ is called its *type*. The *rank* of G is $|I|$. An isomorphism of two combinatorial maps is a type preserving graph isomorphism. *Automorphism* is similarly defined. For $J \subseteq I$ two points of G are *J -adjacent* (J -adj) if they are joined by a path colored in J . Points that are $\{i\}$ -adj are adjacent in the usual sense.

For $J \subseteq I$ let G_J be the subgraph of G obtained by deleting all lines of type not in J . Each connected component of G_J is a combinatorial map over J and is called a *residue* of type J . The only residue of rank $|I|$ is G itself. The residues of rank 0 are the points of G . The residues of type $I - \{i\}$ are called *i -faces* of G . Figure 1 shows a 2-face and a 1-face of the rank 3 combinatorial

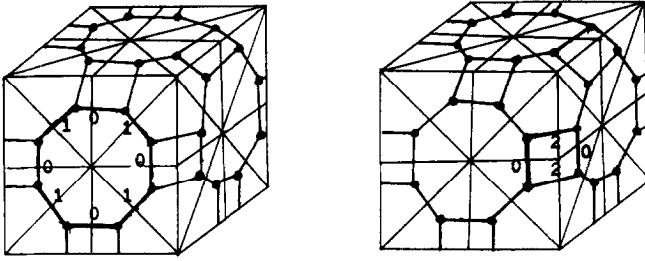


FIG. 1. Faces of the combinatorial map associated with the boundary complex of the cube.

map associated with the cube. Intuitively we have in mind the front 2-face of the cube and its right edge. Two distinct residues R and R' are called *incident* if $R \cap R' \neq \emptyset$. As expected, the 1-face and 2-face in Fig. 1 are incident. Let X denote the set of faces of G ; let $\tau: X \rightarrow I$ be defined by $\tau(x) = i$ if x is an i -face; and let $*$ denote the incidence relation on faces. Then the triple $S(G) = (X, \tau, *)$ is referred to as the *incidence structure of G* .

To any combinatorial map G is associated an $(|I| - 1)$ -dimensional simplicial complex ΔG as follows: For each point v in the point set $V(G)$ of G , let Δv be a simplex of dimension $|I| - 1$. Arbitrarily assign to each vertex of Δv a distinct element of I . Call the set of elements assigned to a face s of Δv the type of s . Let K be the disjoint union of the set $\{\Delta v \mid v \in V(G)\}$. In K identify two simplexes $s \subseteq \Delta v$ and $s' \subseteq \Delta v'$ of the same type J if and only if v and v' are $(I - J)$ -adj. If \sim denotes this identification, take $\Delta G = K/\sim$. Intuitively ΔG can be thought of as being built from $(|I| - 1)$ -simplexes, one for each point of G , such that two $(|I| - 1)$ -simplexes share a common codimension 1 face if the corresponding points are adjacent in G . The space $|G| := |\Delta G|$ is called the *underlying topological space of G* .

EXAMPLE 1. The example of Fig. 1, the combinatorial map of a polyhedron, can be extended. Let K be a cell complex with underlying topological space $|K|$. If $|K|$ is a connected manifold without boundary, then K will be called a *map on a manifold*. In particular, if $|K|$ is a surface, then K is called a *map on a surface*. Given a map K on a manifold, a combinatorial map $G(K)$ is obtained as the dual 1-skeleton of the barycentric subdivision of K . Each vertex of the barycentric subdivision can be labeled with the dimension of the cell it represents. A line of $G(K)$ is then colored i if it joins two maximal simplexes whose labels differ only by i . An i -face of the incidence structure $S(G)$ corresponds to an i -cell of K . Note that G and K have the same underlying topological space. If K is the boundary complex of a polytope P , then the combinatorial map obtained is denoted $G(P)$.

There is exactly one rank 0 and one rank 1 combinatorial map. The classification of rank 2 combinatorial maps is also immediate.

PROPOSITION 2.1. *The rank 2 combinatorial maps are exactly $G(P_n)$, $n \geq 2$, where P_n is an n -gon.*

In Proposition 2.1 we do not rule out the possibility that $n = \infty$. This infinite combinatorial map consists of lines alternately labeled 0 and 1.

3. INCIDENCE STRUCTURES

In the terminology of Tits [16], the incidence structure of a combinatorial map is a thin incidence structure. In this section we explain in what sense thin incidence structures and combinatorial maps are equivalent (Theorem 3.3). First recall some terminology of J. Tits.

Consider a triple $S = (X, \tau, *)$ consisting of a set X , a surjective map $\tau: X \rightarrow I$ and a binary symmetric relation $*$ on X such that for any two elements $x, y \in X$ with $\tau(x) = \tau(y)$, the relation $x*y$ holds if and only if $x = y$. The image by τ of an element or subset of X is called its *type*. The relation $*$ is the *incidence relation*. A *flag* of S is a set of pairwise incident elements of X . A flag is maximal if there is no flag properly containing it. If every maximal flag has cardinality $|I|$, then S is called an *incidence structure over I* . To any incidence structure S over I we can associate a vertex colored abstract simplicial complex called the *chamber complex* $\Delta S = (X, \tau, s)$. The vertex set of ΔS is X . The set s of simplexes of ΔS is the set of flags of S . The maximal simplexes of ΔS are called *chambers*. Both S and ΔS are called *thin* if every simplex of codimension 1 in ΔS is contained in exactly two chambers.

To any thin incidence structure S we can associate a combinatorial map $G(S)$ that is the *dual graph* of S in the following sense: The points of $G(S)$ are the chambers of ΔS and two distinct points u and u' are i -adj in $G(S)$ if and only if u and u' contain a common simplex of type $I - \{i\}$.

EXAMPLE 2. Consider a map K on a manifold. Let X be the set of cells of K and $\tau(x)$ the dimension of a cell x . Call two cells incident if one is contained in the other. Then $S(K) = (X, \tau, *)$ is an *incidence structure*. Under conditions that will be specified in Theorem 3.1, $G(S(K))$ and $G(K)$ are isomorphic.

EXAMPLE 3. A *hypergraph* $H = (Y, E)$ consists of a vertex set Y and a family E of subsets of Y , called *edges*, whose union is Y . If all edges have cardinality 2, then H is a graph. Define a *hypermap* \bar{H} as a two-colored map

on a surface. More precisely $\bar{H} = (K, \theta)$ consists of a map K on a surface and a function $\theta: K_2 \rightarrow \{1, 2\}$ from the set K_2 of 2-cells of K to the two element set $\{1, 2\}$ such that neighboring 2-cells are assigned different values. The underlying hypergraph of \bar{H} is $H = (Y, E)$, where Y is the set of vertices of K and an edge in E is the set of vertices of a 2-cell c in K with $\theta(c) = 1$. In particular, H is a graph if all the 2-cells colored 1 are digons. From a hypermap $\bar{H} = (K, \theta)$ define an incidence structure $S(\bar{H})$ over $I = \{0, 1, 2\}$ as follows: The type 0 elements are the vertices of K ; the type 1 elements are the 2-cells c with $\theta(c) = 1$ and the type 2 elements are the 2-cells c with $\theta(c) = 2$. A type 0 element is incident with a type 1 or 2 element if the vertex lies on the respective 2-cell. A type 1 and type 2 element are incident if the respective 2-cells have an edge in common. Then $G(\bar{H}) := G(S(\bar{H}))$ is a combinatorial map.

As it stands, the correspondence $f: S \mapsto G(S)$, taking thin incidence structures to combinatorial maps is neither one-to-one nor onto. To see that f is not one-to-one consider the 2-dimensional cell complexes K_1 and K_2 in Fig. 2. Both are formed from a triangular prism and two tetrahedra; the interior 2-simplexes are not considered in K_1 and K_2 . If $S_1 = S(K_1)$ and $S_2 = S(K_2)$ are the associated incidence structures, as in Example 2, then $G(S_1) \cong G(S_2)$. This duplicity can be eliminated by removing from consideration certain incidence structures, like $S(K_2)$, that are disconnected in the following sense: A thin incidence structure S is called *residually connected* if the topological link of every simplex in ΔS of codimension > 1 is connected, and the link of every codimension 1 simplex is two vertices. The

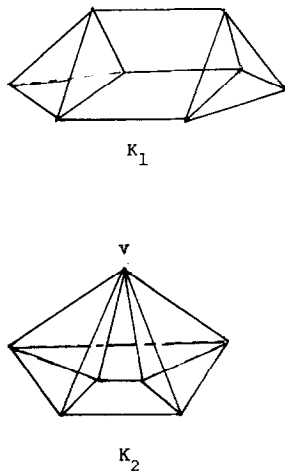


FIG. 2. Incidence structure $S(K_2)$ is not residually connected.

simplicial complex ΔS itself is considered the link of the empty simplex. So S residually connected implies, in particular, that ΔS is connected. Note that S_2 is not residually connected because the link of the vertex v in $\Delta S(K_2)$ is not connected.

To see that $f: S \rightarrow G(S)$ is not onto, let G_0 be the combinatorial map in Fig. 3. Here G_0 is associated, as in Example 1, with the map K on the torus consisting of 2 faces, 4 edges and 2 vertices. It is a consequence of the next theorem that there is no thin incidence structure S_0 such that $G_0 \cong G(S_0)$. A combinatorial map G is called *nondegenerate* if for any finite set $\{R_j\}$ of pairwise incident residues of G , $\cap R_j$ is also a residue of G . Note that G_0 is degenerate, because a residue of type $\{0, 1\}$ and a residue of type $\{1, 2\}$ intersect in two residues of type $\{1\}$.

THEOREM 3.1. *Let G be a combinatorial map. The following statements are equivalent:*

- (1) G is nondegenerate.
- (2) $\Delta G \cong \Delta S(G)$.
- (3) $G \cong G(S_0)$ for some thin, residually connected incidence structure S_0 .

Proof. (1) \Rightarrow (2). Assume statement (1). If s is a simplex in $\Delta S(G)$, then s corresponds to a set $\{R_j\}$ of pairwise incident faces of G . By assumption $\cap R_j$ is a residue in G and hence corresponds to a simplex \hat{s} in G . The assignment $s \mapsto \hat{s}$ induces an isomorphism $\Delta S(G) \rightarrow \Delta G$.

(2) \Rightarrow (3). Assume $\Delta G \cong \Delta S(G)$. Since G is the dual graph of ΔG and $GS(G)$ is the dual graph of $\Delta S(G)$, we have $G \cong GS(G)$. Take $S_0 = S(G)$. By its construction ΔG has connected links, and hence S_0 is residually connected.

(3) \Rightarrow (1). Assume $G \cong G(S_0)$. Consider the assignment $g: R \mapsto s$, from residues of G to simplexes of ΔS_0 , where s is the intersection of the chambers of ΔS_0 corresponding to the points of R . The function g takes residues of type J to simplexes of type $I - J$. Since S_0 is residually

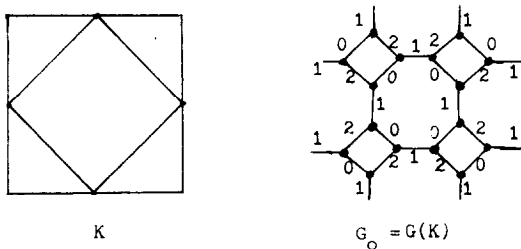


FIG. 3. Degenerate combinatorial map. Opposite sides of K and opposite lines of $G(K)$ are to be identified.

connected, g is a one-to-one correspondence; the inverse is such that the points of R correspond to chambers of ΔS_0 containing s . Now let $\{R_j\}$ be a set of pairwise incident residues of G , and let $s_j = g(R_j)$. The vertices of $\cup s_j$, considered as elements of S_0 , are pairwise incident. Hence they form a simplex s in ΔS_0 . Let $R = g^{-1}(s)$. Now $s_j \subseteq s$ for all j implies $R_j \supseteq R$ for all j implies $\cap R_j \supseteq R$. If L_j is the type of R_j , then both $\cap R_j$ and R have type $\cap L_j$. To conclude that $\cap R_j = R$ it only remains to show that every point of $\cap R_j$ is a point of R . If u is a point of $\cap R_j$, then $g(u) \supseteq s_j$ for all j . Therefore $g(u) \supseteq s$, and x is a point of R . ■

The following result is implicit in the proof of Theorem 3.1:

THEOREM 3.2. *Let G be a nondegenerate combinatorial map over I . There is a one-to-one correspondence g from the set of residues of G to the set of simplexes of ΔG such that for all residues R and R'*

- (1) *type $g(R) = I - \text{type } R$.*
- (2) *$R \subseteq R'$ if and only if $g(R) \supseteq g(R')$.*

We are now in a position to state the appropriate correspondence between combinatorial maps and thin incidence structures.

THEOREM 3.3. *Let \mathcal{G} be the set of nondegenerate combinatorial maps and \mathcal{S} the set of residually connected thin incidence structures. Then the functions $f: \mathcal{G} \rightarrow \mathcal{S}$ and $g: \mathcal{S} \rightarrow \mathcal{G}$ given by $G \mapsto S(G)$ and $S \mapsto G(S)$ are inverse to each other. In particular, there is a one-to-one correspondence between nondegenerate combinatorial maps and thin residually connected incidence structures.*

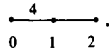
Proof. Let S_0 be a residually connected thin incidence structure and let x be an element of type i in S_0 . The set of all chambers in ΔS_0 containing x corresponds to a residue R_x of type $I - \{i\}$ in $G(S_0)$ and hence to an element \hat{x} of type i in $SG(S_0)$. The assignment $x \mapsto \hat{x}$ induces an isomorphism $S_0 \rightarrow SG(S_0)$.

Conversely, let u be a point of a nondegenerate combinatorial map G_0 . Let $\hat{u} = \{R_i \mid i \in I\}$, where R_i is the residue of type $I - \{i\}$ in G containing u . Then \hat{u} is a point in $GS(G_0)$. Theorem 3.1 implies that the assignment $u \mapsto \hat{u}$ induces an isomorphism $G_0 \rightarrow GS(G_0)$. ■

4. REDUCIBLE AND ORDERED COMBINATORIAL MAPS

Let G be a combinatorial map over I and R a rank 2 residue over $\{i, j\}$. If R is finite, then it is a cycle in G consisting of lines alternately colored i and j . Let $p(R)$ be half the length of this cycle. If R is infinite, then $p(R) = \infty$.

Now let $p_{ij} = \text{lcm}_R p(R)$, where the lcm is taken over all residues of type $\{i, j\}$. The *diagram* $D(G)$ of G is obtained by representing each $i \in I$ as a node labeled i and connecting nodes i and j by a line labeled p_{ij} . By convention the line is omitted when $p_{ij} = 2$ and the line label is omitted when $p_{ij} = 3$. The diagram is a generalization of the Schläfli symbol of a regular polytope. For example, the Schläfli symbol of a cube Q in Fig. 1 is $\{4, 3\}$ indicating that each face is a 4-gon and 3 faces surround each vertex. The diagram of $G(Q)$ is



In general, different combinatorial maps may have the same diagram. The relationship between the diagram of a combinatorial map and the diagram of a Coxeter group is explained in Section 7.

There is a straightforward construction of a rank $(n_1 + n_2)$ combinatorial map from combinatorial maps of rank n_1 and rank n_2 . Let G_1 and G_2 be combinatorial maps over disjoint sets I_1 and I_2 with point sets $V(G_1)$ and $V(G_2)$. The *product* $G_1 * G_2$ is a combinatorial map over $I_1 \cup I_2$ with point set $V(G_1) \times V(G_2)$. Two points (u_1, u_2) and (v_1, v_2) are i -adj whenever $[u_1 = v_1 \text{ and } u_2 \text{ } i\text{-adj } v_2]$ or $[u_2 = v_2 \text{ and } u_1 \text{ } i\text{-adj } v_1]$. This is the standard product construction for graphs, together with the appropriate line coloring. An example is shown in Fig. 4. A combinatorial map is called *reducible* if it is isomorphic to the product of two other combinatorial maps. Otherwise it is *irreducible*.

THEOREM 4.1. *Let G be a combinatorial map. If $D(G)$ is connected, then G is irreducible. Conversely, if G is irreducible and nondegenerate, then $D(G)$ is connected.*

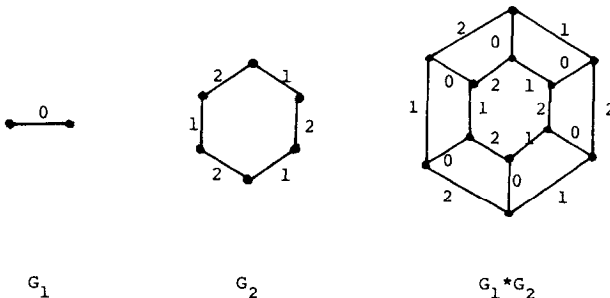


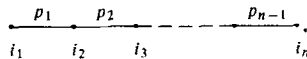
FIG. 4. A reducible combinatorial map.

Proof. Assume G is reducible $G = G_1 * G_2$. It is a consequence of the product construction that $p_{ij} = 2$ for all $i \in I_1$ and $j \in I_2$. Hence $D(G)$ is disconnected.

Conversely, assume that $D(G)$ is disconnected. Let I_1 and I_2 be the nodes in two disjoint components of $D(G)$ with $I_1 \cup I_2 = I$. For any given point u in G let G_1 be the residue of type I_1 containing u and G_2 the residue of type I_2 containing u . We claim that G is isomorphic to $G_1 * G_2$. If $a = \{i_1, i_2, \dots, i_k\}$ is a sequence of elements of I , let \bar{a} denote the terminal point in G of a path with initial point u and successive lines labelled i_1, i_2, \dots, i_k . Because the diagram $D(G)$ is disconnected, each residue of type $\{i, j\}$ with $i \in I_1$ and $j \in I_2$ is a 4-cycle. This implies that for sequences $a \subseteq I_1$ and $b \subseteq I_2$ we have $\overline{ab} = \overline{ba}$. If (u_1, u_2) is an arbitrary point in $G_1 * G_2$, let $a \subseteq I_1$ and $b \subseteq I_2$ be any sequences such that $\bar{a} = u_1$ and $\bar{b} = u_2$. Define a point $f(u_1, u_2)$ in G by $f(u_1, u_2) = \overline{ab}$. Note that $f(u_1, u_2)$ is independent of the choice of a and b ; because if there is another pair of sequences a' and b' with $\bar{a}' = u_1$, $a' \subseteq I_1$ and $\bar{b}' = u_2$, $b' \subseteq I_2$, then $\overline{ab} = \overline{a'b} = \overline{ba'} = \overline{b'a'} = \overline{a'b'}$. The function $f: G_1 * G_2 \rightarrow G$ is an isomorphism. By its definition f preserves i -adjacency. Also f is injective: if $f(u_1, u_2) = f(u'_1, u'_2)$, then there are sequences $a, a' \subseteq I_1$ and $b, b' \subseteq I_2$ with $\bar{a} = u_1$, $\bar{b} = u_2$, $\bar{a}' = u'_1$, $\bar{b}' = u'_2$ and such that $\overline{ab} = \overline{a'b'}$. Letting a^{-1} denote the sequence a in reverse order, this implies $\overline{bb'^{-1}a} = \overline{abb'^{-1}} = \overline{a'b'b'^{-1}} = \overline{a'}$. Hence $\overline{bb'^{-1}} = \overline{a'a^{-1}} \in G_1 \cap G_2 = \{u\}$. The last equality follows from the nondegeneracy of G . Therefore $u_1 = \bar{a} = \bar{a'} = u'_1$ and $u_2 = \bar{b} = \bar{b'} = u'_2$. The surjectivity of f now follows automatically from the connectivity of G . ■

The faces of a polytope are partially ordered by inclusion. In general, an incidence structure $(X, \tau, *)$ is called *ordered* if there is a partial order $>$ on X such that $x * y$ if and only if $x > y$ or $y > x$. A combinatorial map G is called *ordered* if $S(G)$ is ordered. Even rank 3 combinatorial maps exist that are not ordered. Consider, for example, a tessellation of the Euclidean plane by regular congruent hexagons. A 3-coloring of the lines such that each hexagon is 2-colored, yields a rank 3 combinatorial map G , and $S(G)$ is not ordered.

We call a diagram *linear* if it has the form



The p_i are allowed to take the value 2, i.e., the diagram may be disconnected.

THEOREM 4.2. *Let G be a combinatorial map. If $D(G)$ is linear, then G is ordered. Conversely, if G is ordered and nondegenerate, then $D(G)$ is linear.*

Proof. Assume $D(G)$ is linear. Reading $D(G)$ from left to right yields an ordering of $I: i_1 < i_2 < \dots < i_n$. Define an ordering on $S(G)$ as follows: $x < y$ if and only if $x * y$ and $\tau(x) < \tau(y)$. To prove that G is ordered we have only to show that $<$ is transitive. Assume $x < y$ and $y < z$. It is sufficient to show that $x \cap y \neq \emptyset$. Let γ be a path from a point in x to a point in z lying entirely in y and satisfying the conditions: (1) γ is minimal with respect to its length m , and (2) of all paths satisfying (1) γ is minimal with respect to the length m' of the initial subpath with lines labeled $< \tau(y)$. We must show that $m = 0$. Assume $m \neq 0$. If $m' = 0$, then the first line in γ can be deleted, contradicting the minimality of m . Similarly the last line in γ must be labeled $> \tau(y)$. Let u be the first point along γ such that if u' preceeds u and u'' succeeds u along γ , then $\tau(u', u) < \tau(y) < \tau(u, u'')$. Then there is another path $u'vu''$ from u' to u'' such that $\tau(u', v) = \tau(u, u'')$ and $\tau(v, u'') = \tau(u', u)$, contradicting the minimality of m' .

Conversely, assume that G is ordered and nondegenerate. Let $\{x_1, x_2, \dots, x_n\}$ be a maximal flag of $S(G)$ such that $x_1 < x_2 < \dots < x_n$. There is a well-defined total ordering $<$ of I given by $\tau(x_1) < \tau(x_2) < \dots < \tau(x_n)$. Let $i, j \in I$ be nonconsecutive in this order. Let F be a flag of type $I - \{i, j\}$ in $S(G)$. Then F is contained in exactly 4 maximal flags. In terms of $G \cong GS(G)$ this implies that $p_{ij} = 2$. Thus the diagram obtained by positioning nodes $\tau(x_1), \tau(x_2), \dots, \tau(x_n)$ in a row is linear. ■

Let $>$ be an ordering on an ordered combinatorial map G . There is an obvious *dual ordering* $>'$ defined by $x >' y$ if and only if $y > x$.

THEOREM 4.3. *The ordering on an ordered irreducible nondegenerate combinatorial map is unique up to dualization.*

Proof. Assume G is an irreducible nondegenerate combinatorial map with two nondual orders. By Theorem 4.2, the nodes of the diagram $D(G)$ can be ordered in two ways, one not obtainable from the other by merely reversing directions. This implies that $D(G)$ is disconnected, contradicting the assumption that G is irreducible. ■

5. GEOMETRIC REALIZATION

In Examples 1 and 3, combinatorial maps are obtained from maps on surfaces and hypermaps. We now show that any nondegenerate rank 3 combinatorial map can be realized as a hypermap and any ordered rank 3 combinatorial map as a map on a surface.

THEOREM 5.1. *For any nondegenerate rank 3 combinatorial map G there is a hypermap \bar{H} such that $G \cong G(\bar{H})$.*

Proof. Assume that G is a combinatorial map over $I = \{0, 1, 2\}$. Let X_0 be the set of simplexes of $\Delta G = (X, \tau, s)$ of type $\{1, 2\}$. We form an abstract simplicial complex Δ' that is a subdivision of ΔG . The simplicial complex Δ' has vertex set $X' = X \cup X_0$ and the 2-simplexes are triples $\{x, y, z\}$, where $\tau(x) = 0$ and $z = \{y, w\} \in X_0$ with $x*y$ and $x*w$. The cells of a map on a surface K are now formed from the unions of simplexes of Δ' as follows: For a vertex $x \in X_0$ let $c_1(x)$ be the union of the two edges of Δ' containing x and a vertex of type 0. Take $\{c_1(x) \mid x \in X_0\}$ as the set of 1-cells of K . There are two kinds of 2-cells. For a vertex x of Δ' let $c_2(x)$ be the union of the 2-simplexes of Δ' containing x . The set of 2-cells of K with θ -value 1 are $\{c_2(x) \mid \tau(x) = 1\}$, and those with θ -value 2 are $\{c_2(x) \mid \tau(x) = 2\}$. Then (K, θ) is the desired hypermap. ■

THEOREM 5.2. *For any ordered rank 3 combinatorial map G there is a map K on a surface such that $G \cong G(K)$.*

Proof. There is no loss of generality in assuming that G is a combinatorial map over $\{0, 1, 2\}$. The simplicial complex $\Delta G = (X, \tau, s)$ is a 2-dimensional pseudomanifold and hence ΔG is a surface. For $x \in X$ let $c(x)$ denote the subcomplex of ΔG consisting of those simplexes whose vertices are incident with x and are of type $\leq \tau(x)$. Since G is ordered, $c(x)$ is a pseudomanifold of dimension $\tau(x)$. Let $|c(x)|$ denote the union of the simplexes of $c(x)$. Let K be the map with underlying surface $|G|$, where the set of i -cells of K is $\{|c(x)| \mid \tau(x) = i\}$. Then $G \cong G(K)$. ■

Theorem 5.2 does not extend to combinatorial maps of rank >3 . As in Example 1, every map on a manifold yields a combinatorial map. However, not all ordered combinatorial maps arise in this way. In general, $|G|$ need not be a manifold at all. Consider

EXAMPLE 4. Let $L_n = \{0, 1, 2, \dots, n\}$ and K_{n+1} an abstract simplicial complex whose vertex set is $L_n \times L_{n+1}$ and whose simplexes are the sets of the form $\{(0, m_0), (1, m_1), \dots, (n, m_n)\}$ and all nonempty subsets, where the m_i are distinct. The spaces $|K_2|$ and $|K_3|$ are homeomorphic to a 1-sphere and a torus, respectively. It is easy to verify that the link of any vertex of K_n is isomorphic to K_{n-1} . Hence the link of each vertex of K_4 is a torus. If $G = G(K_4)$, then $|G| \cong |K_4|$ is not a manifold.

6. TOPOLOGICAL CONCEPTS

For a combinatorial map, properties of the underlying topological space $|G|$ are related to the combinatorial properties of G . In this section orientability and the fundamental group are discussed.

THEOREM 6.1. *Let G be a combinatorial map. Then $|G|$ is orientable if and only if G is a bipartite graph.*

Proof. Without loss of generality assume that $I = \{0, 1, \dots, n\}$. A collection $\{\sigma(c)\}$ of orientations of the n -simplexes of ΔG is called compatible (see [11]) if for any $(n-1)$ -simplex s which is a face of n -simplexes c and c' the orientations $\sigma(c)$ and $-\sigma(c')$ induce the same orientation on s . The space $|G|$ is orientable if and only if there exists such a compatible collection of orientations. There is a natural orientation on each chamber of G given by labeling each vertex with its type. These local orientations can be altered in sign to be made compatible if and only if all be cycles in G have even length, i.e., if and only if G is bipartite. ■

A path in a combinatorial map G is a sequence of points $\{u_1, u_2, \dots, u_n\}$ such that consecutive points are adjacent. The following notion of m -homotopy is due to Tits [17]. If α and β are paths such that the last point in α is the first point in β , then the product $\alpha\beta$ is the concatenated path. The inverse α^{-1} is the path obtained by listing the points of α in reverse order. Two paths $\alpha = \beta\gamma\delta$ and $\alpha' = \alpha'\gamma'\delta$ are called elementary m -homotopic ($\alpha \sim \alpha'$) if γ and γ' are contained in some residue of rank m . Two paths α and α' are m -homotopic if there is a sequence of elementary m -homotopies $\alpha = \alpha_1 \sim \alpha_2 \sim \dots \sim \alpha_k = \alpha'$ such that all the α_i have the same first and last elements. If u_0 is a fixed base point of G , the set of m -homotopy classes of closed path based at u_0 forms a group in the usual way. This group, which is independent of the base point, is called the h -homotopy group of G and is denoted by $\pi^m(G)$. The relation between this combinatorial notion and the fundamental group $\pi(|G|)$ of the underlying topological space is the subject of the next result.

THEOREM 6.2. *If G is a rank n combinatorial map, then $\pi(|G|) \cong \pi^{n-1}(G)$.*

Proof. Define a homomorphism $\Theta: \pi^{n-1}(G) \rightarrow \pi(|G|)$ as follows: A closed path α in G is mapped to the closed path $\Theta\alpha$ in $|G|$ formed by joining the barycenters of the chambers of ΔG corresponding to consecutive points of α via the common codimension 1 simplex. In this proof \sim denotes $(n-1)$ -homotopy in G and \simeq denotes topological homotopy in $|G|$. To show that Θ is a welldefined isomorphism it is sufficient to prove that Θ is surjective and that $\alpha \sim 0$ if and only if $\Theta\alpha \simeq 0$. Let α be a closed path in $|G|$ based at an interior point of some chamber. By a general position argument there is a path α' such that $\alpha \simeq \alpha'$ and α' does not pass through any simplex of codimension > 1 . This is sufficient to show that Θ is surjective.

Assume $\alpha \sim 0$. To prove $\Theta\alpha \simeq 0$ it is sufficient to show that if $\alpha \sim \beta$ is an

elementary $(n-1)$ -homotopy, then $\Theta\alpha \simeq \Theta\beta$. But $\Theta\alpha$ and $\Theta\beta$ differ only on the open star of some vertex of $|\Delta G|$. Since such a subspace is contractible $\Theta\alpha \simeq \Theta\beta$. Conversely, assume that $\Theta\alpha \simeq 0$. Let $F: [0, 1] \times [0, 1] \rightarrow |G|$ be the homotopy from $\Theta\alpha$ to 0 so that $F(0, s) = \Theta\alpha(s)$ and $F(1, s) = 0$. We may assume by standard topological arguments that there are sequences of numbers $0 < t_1 < t_2 < \dots < t_m < 1$ and s_1, s_2, \dots, s_m such that $F(t, s)$ does not lie on a simplex of codimension 1 unless $(t, s) = (t_i, s_i)$ for some i . Let $t_i < \hat{t}_i < t_{i+1}$ and let α_{t_i} be the path in G corresponding to $F(t_i, \cdot)$. If $F(t_i, s_i)$ lies on a simplex σ of codimension > 1 , then $\alpha_{t_{i-1}}$ and α_{t_i} differ only on the residue of rank $< n$ corresponding to the chambers of ΔG containing σ . Therefore $\alpha_{t_{i-1}}$ and α_{t_i} are elementary $(n-1)$ -homotopic and $\alpha \sim 0$. ■

COROLLARY 6.3. *Let G be a combinatorial map. If the underlying topological space $|G|$ is a simply connected manifold, then $\pi^2(G) = 0$.*

Proof. Since G is a simply connected manifold, each open star $\text{st}(s)$ is contractible, hence simply connected, when s is a simplex of codimension > 2 in ΔG . Applying the argument in the proof of Theorem 6.2 to residues, any two paths in a residue R of rank $m > 2$ with the same initial and end points are $(m-1)$ -homotopic in R . If α is a closed path in G , then α is $(n-1)$ -homotopic to 0, where $n = \text{rank } G$. But for $m > 2$ any sequence of elementary m -homotopies $\alpha = \alpha_1 \sim \alpha_2 \sim \dots \sim \alpha_k = 0$ can be refined to a sequence of elementary $(m-1)$ -homotopies $\alpha = \alpha'_1 \sim \alpha'_2 \sim \dots \sim \alpha'_{k'} = 0$. Therefore α is 2-homotopic to 0. ■

Ramified coverings have been studied by both Tits [17] and Ronan [13] in a more general setting. We repeat the definitions in the present context. Let G and G' be maps over I . For a nonnegative integer m , an m -covering $G' \rightarrow G$ is a function $f: V(G') \rightarrow V(G)$ that preserves i -adjacency for all $i \in I$ and is bijective when restricted to rank m residues. An m -covering is automatically an $(m-1)$ -covering. An $|I|$ -covering is an isomorphism. By a *covering* we mean an m -covering for some $m \geq 0$. The covering f naturally induces a topological map $|f|: |G'| \rightarrow |G|$. An $(|I|-1)$ -covering induces a topological covering of the underlying topological spaces. The integer m can be thought of as a measure of the ramification of the covering. For $u \in V(G)$ the set $f^{-1}(u)$ is called the *fiber* above u . It is easy to show that any two fibers have the same cardinality. If this cardinality is d , we say that f is a d -fold covering. The group of automorphisms of G' preserving each fiber is called the *group of covering transformations* of f . Two coverings $f: G'_1 \rightarrow G_1$ and $g: G'_2 \rightarrow G_2$ are called *equivalent* if there exist isomorphisms $\theta: G'_1 \rightarrow G'_2$ and $\phi: G_1 \rightarrow G_2$ such that $\phi \circ f = g \circ \theta$.

An m -covering $\hat{G} \rightarrow G$ is called a *universal m -covering* if it possesses the

following universal property: If $g: G' \rightarrow G$ is any other m -covering, then f factors through g :

$$\begin{array}{ccc} & \hat{G} & \\ \swarrow & \downarrow f & \\ G' & \xrightarrow{g} & G \end{array}$$

For example, consider the map K on a torus obtained by identifying opposite sides of the square in Fig. 5. The universal 2-cover of $G(K)$ is $G(\hat{K})$, where \hat{K} is the infinite tessellation of the Euclidean plane into squares. It is clear that if G possesses a universal m -covering, then it is unique. The existence of a universal m -covering of G is shown as in topological covering space theory: Let G be a combinatorial map over I with base point u and $0 \leq m \leq \text{rank } G$. Let \hat{G} be the I -labeled graph whose points are the m -homotopy classes of paths with initial point u_0 in G . Two points α and β are i -adj in \hat{G} if and only if their endpoints are i -adj in G . The universal m -covering $f: \hat{G} \rightarrow G$ is obtained by mapping each point α of \hat{G} to its endpoint in G . A combinatorial map G is called *simply m -connected* if G is its own universal m -cover. A simply $(|I| - 1)$ -connected combinatorial map is called *simply connected*. By Theorem 6.2 and Proposition 6.4, simple connectivity corresponds to the usual topological notion. The first part of the proposition follows immediately from the construction of the universal m -covering. The second part has essentially the same proof as Corollary 6.3 and we omit it.

PROPOSITION 6.4. *Let G be a combinatorial map.*

- (1) *G is simply m -connected if and only if $\pi^m(G) = 0$.*
- (2) *If all residues of rank $> m$ in G are simply connected, then G is simply m -connected.*

An intuitive idea for constructing a d -fold m -covering f of a combinatorial map G is to stack d chambers above each chamber in ΔG and "glue"

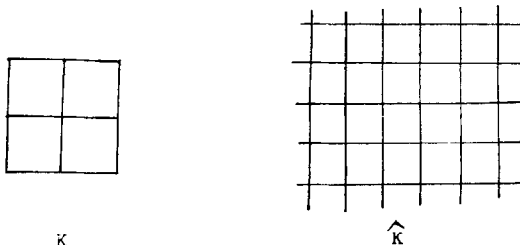


FIG. 5. $G(\hat{K})$ is the universal 2-cover of $G(K)$.

together pairs of chambers in adjacent stacks according to rules that insure that the natural projection onto ΔG induces an m -covering. We now formulate this idea rigorously. It will be applied to the construction of highly symmetric maps in [20].

Let Σ_d denote the symmetric group acting on the set $D = \{1, 2, \dots, d\}$. A permutation representation of a group π in Σ_d is a homomorphism $f_*: \pi \rightarrow \Sigma_d$. Two such permutation representations $f_*, f'_*: \pi \rightarrow \Sigma_d$ are *equivalent* if there is a permutation $\sigma \in \Sigma_d$ such that $\sigma \circ f_* \alpha = f'_* \alpha \circ \sigma$ for all $\alpha \in \pi$. A permutation representation is called *transitive* if $f_*(\pi)$ acts transitively on D .

We now set up a correspondence between m -coverings of G and transitive permutation representations of the homotopy group $\pi^m(G)$. Let $f: G' \rightarrow G$ be a d -fold m -covering. Let u_0 be a fixed base point of G and denote by $D = \{1, 2, \dots, d\}$ the set of points in the fiber of f lying above u_0 . For $s \in D$ any closed path α based at u_0 has a unique lifting to a path α' in G' with initial point s . Let t be the terminal point of α' and let $f_*(\alpha): D \rightarrow D$ be defined by $s \mapsto t$. The permutation representation $f_*: \pi^m(G) \rightarrow \Sigma_d$ is well defined up to equivalence. The connectedness of the graph G' implies the transitivity of f_* . Hence we have a function $\Phi: f \mapsto f_*$ from d -fold m -coverings to transitive permutation representations of $\pi^m(G)$ in Σ_d .

In the other direction let $f_*: \pi^m(G) \rightarrow \Sigma_d$ be a transitive permutation representation of $\pi^m(G)$. Let T be a spanning tree of G . Construct a combinatorial map G' with point set $V(G) \times D$. Two points (u, r) and (u', r') of G' are declared i -adj if u and u' are i -adj and either (1) $(u, u') \in T$ and $r = r'$ or (2) $(u, u') \notin T$ and $r' = (f_* \alpha) r$, where α is the unique cycle in $T \cup \{u, u'\}$ containing the ordered pair (u, u') . Let $f: G' \rightarrow G$ be the covering defined by $(u, r) \mapsto u$. It is not difficult to show that f is an m -covering and independent of the choice of base point. The independence of the choice of the spanning tree T is part of the proof of Theorem 6.5. Thus we obtain a function $\Phi_*: f_* \mapsto f$ from transitive permutation representations of $\pi^m(G)$ in Σ_d to d -fold m -coverings of G .

THEOREM 6.5. *For a combinatorial map G the functions Φ and Φ_* defined above are inverse to each other. In particular, the d -fold m -coverings of G are in one-to-one correspondence with the transitive permutation representations of $\pi^m(G)$ in Σ_d .*

Proof. We first show that for a given f_* the corresponding f is independent of the spanning tree. For the covering $f: G' \rightarrow G$ constructed above define functions $\Phi_i: V(G) \rightarrow \Sigma_d$ for all $i \in I$ as follows: $(\Phi_i u) r = r'$ if (u, r) is i -adj to (u', r') in G' for some u' . Now let u_0 be the base point of G and α a path $\{u_0, u_1, \dots, u_k = u\}$ from u_0 to u in G , where u_{j-1} is i_j -adj to u_j . Let $\Phi(\alpha)$ be the permutation $\Phi_{i_k} u_{k-1} \circ \Phi_{i_{k-1}} u_{k-2} \circ \dots \circ \Phi_{i_1} u_0$. If α is a

closed path, then $\Phi\alpha$ depends only on the m -homotopy class of α . This is because $\Phi\alpha = f_*\alpha$ in this case. Now let \hat{T} be another spanning tree of G with corresponding covering $\hat{f}: \hat{G} \rightarrow G$ and corresponding function $\hat{\Phi}$. If α and β are any two paths from u_0 to u , then $\Phi\alpha\beta^{-1} = \hat{\Phi}\alpha\beta^{-1}$ implies $\hat{\Phi}\alpha \circ \Phi^{-1}\alpha = \hat{\Phi}\beta \circ \Phi^{-1}\beta$. Therefore the function $\hat{\Phi} \circ \Phi^{-1}: V(G) \rightarrow \Sigma_d$ is independent of the path. The function $g: G' \rightarrow \hat{G}$ given by $(u, r) \mapsto (u, (\hat{\Phi} \circ \Phi^{-1}u)r)$ is an isomorphism. Since $\hat{f} \circ g = f$, the coverings f and \hat{f} are equivalent.

To show that $\Phi_* \circ \Phi = id$, let $f: G' \rightarrow G$ be a covering; let $f_*: \pi^m(G) \rightarrow \Sigma_d$ be the corresponding permutation representation and let $\hat{f} = \Phi_* f_*: \hat{G} \rightarrow G$. Chose a base point u_0 in G . In the process of constructing f_* we gave the numeral 1 to a point in the fiber of f above u_0 . Let u'_0 be this point. Let u' be an arbitrary point of G' and α' a path from u'_0 to u' in G' . Let $\hat{\alpha}$ be the unique lifting of $f(\alpha')$ for the covering $f: \hat{G} \rightarrow G$ starting at the point $(u, 1)$. Let \hat{u} be the end point of $\hat{\alpha}$. If $g: G' \rightarrow \hat{G}$ is the isomorphism given by $u' \mapsto \hat{u}$ we have the equivalence $f = \hat{f} \circ g$. In the opposite direction $\Phi \circ \Phi_* = id$ follows directly from the definitions. ■

7. SCHREIER REPRESENTATION

In this section we discuss a group theoretic representation of a combinatorial map. Let W be a group generated by involutions $\{r_i \mid i \in I\}$. By abuse of language we often use the letter W to indicate both the group and the distinguished set of involutions. If H is a subgroup of W the *Schreier coset graph* $G(W, H)$ is an I -labeled graph defined as follows: The points of $G(W, H)$ are the right cosets of W/H and two points u and u' are i -adj if and only if $u' = ur_i$. When H is the trivial subgroup of W , the Schreier coset graph is the Cayley graph of W with respect to the generators $\{r_i \mid i \in I\}$. The graph $G(W, H)$ is a combinatorial map over I . For example, let $W = \langle r_0, r_1, r_2 \mid r_0^2 = r_1^2 = r_2^2 = (r_0 r_1)^4 = (r_1 r_2)^4 = (r_0 r_2)^2 = 1 \rangle$ and $H = \langle (r_0 r_1 r_2)^2, (r_1 r_2 r_0)^2 \rangle$. Then $G(W, H) = G_0$, where G_0 is the combinatorial map in Fig. 3.

Let G be any combinatorial map over I . For each $i \in I$ define a permutation ρ_i of $V(G)$ by $\rho_i u = u'$ if u i -adj u' . Let P be the permutation group on $V(G)$ generated by the $\{\rho_i\}$ and let P_u be the stabilizer of a point u of G . The function $G(P, P_u) \rightarrow G$ given by $P_u \mapsto g^{-1}(u)$ is an isomorphism, yielding the following result:

THEOREM 7.1. *If G is a combinatorial map, then $G = G(W, H)$ for some group W generated by involutions and subgroup $H \leq W$.*

If $G = G(W, H)$ then $G(W, H)$ will be called a *Schreier representation* of the combinatorial map G . By the previous result, every combinatorial map

has a Schreier representation. According to the next result the Schreier representation is essentially unique. Recall that the core of H in W is the largest subgroup of H normal in W . Also \sim denotes conjugacy and $H^g = g^{-1}Hg$.

THEOREM 7.2. *Let W and W' be groups generated by involutions $\{r_i \mid i \in I\}$ and $\{r'_i \mid i \in I\}$ respectively, and let $H \leq W$ and $H' \leq W'$. Then $G(W, H) \cong G(W', H')$ if and only if the bijection $r_i \mapsto r'_i$ induces an isomorphism $\phi: W/N \rightarrow W'/N'$ and $\phi(H/N) \sim H'/N'$, where N and N' are the cores of H in W and H' in W' .*

Proof. It is easily shown that $G(W/N, H/N) \cong G(W, H)$ and $G(W'/N', H'/N') \cong G(W', H')$. Hence we may assume without loss of generality that N and N' are the trivial subgroups. Assume $\phi: W \rightarrow W'$ is an isomorphism and $\phi(H) \equiv H'^g$. Then the function $Ha \mapsto H'g\phi(a)$ induces an isomorphism $G(W, H) \cong G(W', H')$. Conversely, let $f: G(W, H) \rightarrow G(W', H')$ be an isomorphism and say $f(H) = H'g$. Let \hat{H} be the subgroup of W consisting of all finite products $\prod r_{i_j}$ such that $\prod r'_{i_j} = 1$ in W' . Define \hat{H}' in the same way by reversing the roles of r_i and r'_i . Note that $\hat{H} \leq H$ and $\hat{H} \trianglelefteq W$. But $\hat{H} \leq N = \{1\}$ implies that $\hat{H} = \{1\}$. In the same way $\hat{H}' = \{1\}$. This is sufficient for $r_i \mapsto r'_i$ to induce an isomorphism $\phi: W \rightarrow W'$. Since $f(H) = H'g$ we have $Ha = H$ if and only if $H'g\phi(a) = H'g$. Therefore $\phi(H) = H'^g$. ■

COROLLARY 7.3. *If W is a group generated by involutions with subgroups H and H' , then $G(W, H) \cong G(W, H')$ if and only if $H \sim H'$.*

There are two particularly useful Schreier representations of a combinatorial map—the Coxeter representation and the canonical representation. A Coxeter group over I is a group generated by involutions with the presentation

$$W = \langle r_i, i \in I \mid (r_i r_j)^{p_{ij}} = 1, p_{ii} = 1, p_{ij} \geq 2 \rangle. \quad (7.1)$$

By abuse of language, W refers to presentation (7.1) as well as the group. We do not eliminate the possibility that $p_{ij} = \infty$ in which case the relation $(r_i r_j)^{p_{ij}} = 1$ is absent. The *diagram* for a Coxeter group is constructed by representing r_i by a node labeled i and connecting nodes i and j by a line labeled p_{ij} . By the usual convention the line is omitted when $p_{ij} = 2$, and the line label is omitted when $p_{ij} = 3$. The significance of a disconnected diagram D is that W is the direct product of the subgroups generated by the involutions corresponding to the nodes of each connected component of D . A Coxeter group is therefore said to be *irreducible* if its diagram is connected. Coxeter [2] classified all finite irreducible groups with presentation (7.1). These will be discussed in detail in relation to regular combinatorial maps [20].

If W is a Coxeter group it is apparent that the diagram of the combinatorial map $G(W, \{1\})$ is the same as the diagram of W . For example, if W is the Coxeter group with diagram \xrightarrow{r} , then for $r=3$, $G(W, \{1\})$ is a tetrahedron, i.e., $G(W, \{1\}) \cong G(P)$, where P is a tetrahedron. If $r=4$, then $G(W, \{1\})$ is a cube. If $r=5$, then $G(W, \{1\})$ is a dodecahedron. If $r=6$, then $G(W, \{1\})$ is the tessellation of the Euclidean plane into regular hexagons. If $r>6$, then $G(W, \{1\})$ is the tessellation of the hyperbolic plane (open unit disk) into regular r -gons, 3 of them surrounding each vertex.

Let G be a combinatorial map over I . Let P be the permutation group on $V(G)$, with generators $\{\rho_i\}$, described at the beginning of this section. If W is any group generated by involutions $\{r_i\}$ and $\phi: W \rightarrow P$ is a homomorphism induced by $r_i \mapsto \rho_i$, then $G(W, \phi^{-1}(P_u)) \cong G(P, P_u) \cong G$. By taking W to be the Coxeter group of form (7.1) with diagram $D(G)$, we obtain the following result:

COROLLARY 7.4. *Every combinatorial map G has a Schreier representation $G(W, H)$, where W is a Coxeter group with the same diagram as G .*

The representation of Corollary 7.4 is the *Coxeter representation*. If we take $W_* = \langle r_i, i \in I \mid r_i^2 = 1 \rangle$, then $G(W_*, \phi^{-1}(P_u))$ is called the *canonical representation* of G .

We conclude this section by expressing several properties of a combinatorial map in terms of the Schreier representation. Let $\Gamma(G)$ denote the automorphism group of the combinatorial map G as defined in Section 2. Also $N_w(H)$ denotes the normalizer of the subgroup H in W .

THEOREM 7.5. *If G is a combinatorial map with Schreier representation $G(W, H)$, then $\Gamma(G) \cong N_w(H)/H$. Moreover $\Gamma(G)$ acts transitively on $V(G)$ if and only if $H \trianglelefteq W$.*

Proof. For each $a \in N_w(H)$ the function $f_a: Hb \rightarrow Hab$ induces an automorphism of $G(W, H)$. Hence there is a homomorphism $\phi: N_w(H) \rightarrow \Gamma(G(W, H))$ given by $a \mapsto f_a$. Since $\ker \phi = H$ we have only to show that ϕ is surjective. Let $f \in \Gamma(G(W, H))$ and assume that $f(H) = Ha$. This implies that $f(Hb) = Hab$ for all $b \in W$. Therefore $f = f_a$. Moreover, $a \in N_w(H)$ because $Hah = f(Hh) = f(H) = Ha$ for all $h \in H$.

The automorphism group $\Gamma(G)$ is transitive on $V(G)$ if and only if for all $a \in W$ there is an $n \in N_w(H)$ such that $Ha = Hn$, i.e., $a \in N_w(H)$. But this is equivalent to $W = N_w(H)$. ■

THEOREM 7.6. *Let $f: G' \rightarrow G$ be a covering of combinatorial maps with group T of covering transformations. There is an equivalent covering $f: G(W, H') \rightarrow G(W, H)$ with $H' \leq H$ defined by $f(H'a) = Ha$. Moreover*

- (1) f is an $(H: H')$ -fold covering.
- (2) $T \cong N_H(H')/H'$.
- (3) T acts transitively on each fiber if and only if $H' \trianglelefteq H$.

Proof. With notation as above let $G(W_*, H_*)$ be the canonical representation of G' and $G(W_*, H_*)$ the canonical representation of G . If $f(H'_*) = H_* g$, take $W = W_*$, $H' = H'_*$, and $H = H_*$. Then $H' \leq H$ and the covering $f: G(W, H') \rightarrow G(W, H)$ defined by $\hat{f}(H'a) = Ha$ is equivalent to f . Statement (1) follows immediately. To prove (2) consider the homomorphism $\phi: N_W(H) \rightarrow \Gamma(G(W, H'))$ defined by $(\phi a)(H'b) = H'ab$. Then $\phi a \in T$ if and only if $\hat{f} \circ \phi a = \hat{f}$, which occurs exactly when $a \in H$. Hence we have a surjective homomorphism $\phi': N_H(H') \rightarrow T$ and $\ker \phi' = H'$. To show (3) let $h \in H$. If T acts transitively on each fiber, then there exists an automorphism taking H' to $H'h$, i.e., there exists an $a \in N_H(H')$ such that $H'a = H'h$. Therefore $h \in H'a \leq N_H(H')$. Conversely, assume $H' \trianglelefteq H$ and let $H'a$ and $H'b$ lie in the same fiber. Then $ba^{-1} \in H = N_H(H')$. The automorphism $\phi(ba^{-1})$ takes Ha to Hb . ■

Let G be a combinatorial map with canonical representation $G(W, H)$. For $J \subseteq I$ let W_J be the subgroup of W generated by $\{r_i \mid i \in J\}$ and let H_m be the subgroup of H generated by the subgroups $\{W_J^a \cap H \mid a \in W \text{ and } |J| = m\}$.

THEOREM 7.7. *Let G be a combinatorial map with canonical Schreier representation $G(W, H)$. With the notation as above $f_m: G(W, H_m) \rightarrow G(W, H)$ defined by $f_m(H_m a) = Ha$ is equivalent to the universal m -covering of G .*

Proof. We first show that f_m is an m -covering. Assume that $H_m a$ and $H_m b$ are two points in the same residue of type J , $|J| = m$, and $f_m(H_m a) = f_m(H_m b)$. That $Ha = f_m(H_m a) = f_m(H_m b) = Hb$ implies $ab^{-1} \in H$. That $H_m a$ and $H_m b$ are in the same residue of type J implies that there exists an $x \in W_J$ such that $H_m ax = H_m b$, i.e., $axb^{-1} \in H_m \leq H$. Since $ab^{-1} \in H$ and $axb^{-1} \in H$ we have $axa^{-1} \in H \cap W_J^a \subseteq H_m$. Since $axa^{-1} \in H_m$ and $axb^{-1} \in H_m$ we have $ab^{-1} = (axa^{-1})^{-1}(axb^{-1}) \in H_m$. Therefore $H_m a = H_m b$.

Now assume that $f: G(W, H') \rightarrow G(W, H)$ is any other m -covering of G where we may assume without loss of generality that $H' \leq H$ and $f(H'a) = Ha$ for all $a \in W$. We claim that $H_m \leq H'$. It is sufficient to show that any element of $W_J^a \cap H$ belongs to H' . So assume $awa^{-1} \in W_J^a \cap H$, where $w \in W_J$. Then $Hawa^{-1} = H$ implies $Haw = Ha$. Since $H'aw$ and $H'a$ lie in the same residue of type J and f is bijective on residues of type J , $H'aw = H'a$. Therefore $awa^{-1} \in H'$. Now if $g: G(W, H_m) \rightarrow G(W, H')$ is defined by $g(H_m a) = H'a$ for all $a \in W$, then $f \circ g = f_m$. ■

THEOREM 7.8. *If G is a combinatorial map with canonical representation $G(W, H)$, then $\pi^m(G) \cong H/H_m$ for $m \geq 1$.*

Proof. Regarding G as a Schreier coset graph there is a well-defined homomorphism $\phi: H \rightarrow \pi^m(G)$ defined by $\phi(r_{i_1} r_{i_2} \cdots r_{i_k}) = \{1, r_{i_1}, r_{i_1} r_{i_2}, \dots, r_{i_1} r_{i_2} \cdots r_{i_k}\}$. The subgroups $H \cap W_J^a$ with $|J| = m$ generate H_m . Since $H \cap W_J^a \subseteq \ker \phi$ we have $H_m \subseteq \ker \phi$. The converse, that $h \in \ker \phi \Rightarrow h \in H_m$, is proved by an easy induction on the length of the chain $\phi(h) = \alpha_1 \sim \alpha_2 \sim \cdots \sim \alpha_k = 0$. ■

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