CONTRACTIVE DIGRAPHS AND SPLICING MACHINES

ANDREW VINCE

ABSTRACT. Contractive digraphs and splicing machines, notions that seem to be new in the theory of directed graphs, are introduced. The existence and uniqueness of the central object, the attractor of a contractive digraph, is proved, and the attractor is shown to be the unique fixed point of a natural operator defined on sets of vertices of the digraph. The attractor of a contractive digraph is then shown to be splicing machine. Motivation comes, on the theoretical side, from the Banach fixed point theorem, and, on the applied side, from the splicing of sequences from a finite alphabet.

1. INTRODUCTION

Let $G$ be a finite, directed graph on vertex set $V$. An edge in $G$, directed from vertex $x$ to vertex $y$ is denoted $(x,y)$. A path is always a directed path. The length of a path $p$, if finite, is the number of edges in $p$. A circuit is a closed path. A cycle is a circuit with no repeated vertices (except the first and the last); i.e., a cycle does not cross itself.

The main objects in this paper are contractive digraphs and splicing machines, the definitions given below. Motivation comes, on the theoretical side, from the Banach fixed point theorem, and, on the applied side, from the splicing of sequences from a finite alphabet. The two concepts, contractive digraphs and splicing machines, turn out to be closely connected.

Figure 1 shows a 2-colored (black and red) digraph with the property that each vertex has exactly one outgoing edge colored black and exactly one outgoing edge colored red. (There are loops at vertices 1 and 8.) If the successive colors along a path $p$ are $(c_1,c_2,c_3,\ldots,c_n)$, then we say that $p$ has type $(c_1,c_2,c_3,\ldots,c_n)$. Consider a sequence of colors, say $C = (1,0,0,0,0,1,1,0,1,0,1,1,0,0)$, where 0 stands for black and 1 for red. In the figure, the circuit with successive vertices 2, 5, 3, 2, 1, 1, 5, 7, 4, 6, 3, 6, 7, 8, 4, 2 is of type $C$. If fact, this particular digraph has the following property: (1) for any binary sequence $C$ of colors, no matter how long, there is a circuit in the digraph of type $C$; (2) for any sequence $C$ of colors, the circuit in the digraph of type $C$ is unique; and (3) there are no “extra” edges in the digraph in the sense that every edge appears in some circuit. Since every circuit in a digraph can be obtained by “splicing” cycles together, we will refer to such a digraph as a splicing machine.

\begin{figure}[h]
\centering
\includegraphics{splicing-machine.png}
\caption{A splicing machine.}
\end{figure}

\textbf{2010 Mathematics Subject Classification.} 05C20.
\textbf{Key words and phrases.} digraph, splicing.
defined formally in Definition 3 below. Basically, in a splicing machine, any circular sequence of colors can be unique obtained by splicing together a subset of the finitely many cycle sequences.

1.1. Contractive Digraphs.

Definition 1. Let \([N] = \{1, 2, \ldots, N\}\) for \(N \geq 1\), and call \([N]\) the set of colors. A colored-digraph \(G = (V, E, c)\) is a finite directed graph with vertex set \(V\), edge set \(E\), and edge coloring \(c : E \rightarrow [N]\) such that every vertex has exactly \(N\) incident edges directed out, one outward edge of each color 1, 2, \ldots, \(N\). Multiple edges and loops are allowed.

For a colored-digraph \(G\) whose edges are colored in \([N]\) and a path \(p = x_0, x_1, \ldots\), finite or infinite, the type of \(p\), denoted \(C_p\), is defined as

\[
C_p = (c(x_0, x_1), c(x_1, x_2), c(x_2, x_3)\ldots).
\]

Given a sequence \(C = (j_1, j_2, \ldots)\), finite or infinite, of colors, and a vertex \(x_0 \in V\), there is a unique path, denoted \(p_C(x_0)\), of type \(C\). The same vertex may, of course, appear many times in \(p_C(x_0)\).

If an infinite path \(p\) has successive vertices \(x_0, x_1, x_2, \ldots\) and an infinite path \(p'\) has successive vertices \(x'_0, x'_1, x'_2, \ldots\), then we say that \(p\) and \(p'\) are parallel if \(x_i \neq x'_i\) for all \(i \geq 0\). Let \([N]^*\) denote the set of all finite sequences of colors and \([N]^{\infty}\) the set of all infinite sequences of colors. Given a sequence \(C \in [N]^{\infty}\), parallel paths \(p_C(x_0) = x_0, x_1, \ldots\) and \(p_C(y_0) = y_0, y_1, \ldots\), with the same color sequence \(C \in [N]^{\infty}\), will be called \(C\)-parallel.

Definition 2. A colored-digraph \(G\) is called contractive if \(G\) has no pair of \(C\)-parallel paths for all \(C \in [N]^{\infty}\). Such a colored-digraph will be referred to as a contractive digraph.

![Small contractive digraphs with two colors.](image)

Figure 2. Small contractive digraphs with two colors.

Note that, if \(G\) is contractive, then it must be connected as an undirected graph. Four small contractive digraphs with \(N = 2\) are shown in Figure 2. Several infinite families of contractive digraphs are provided in the examples below. The terminology “Cantor set” and “Sierpinski triangle” in Examples 2 and 3 will be explained in Example 6 of Section 2. The examples below are revisited in Example 7 and Example 8.

Example 1 (Discrete Interval). Consider the following infinite family \(G(2M)\) for \(M = 1, 2, \ldots\), of 2-colored-digraphs. Let \(V = \{0, 1, 2, \ldots, 2M - 1\}\). The edges colored 1 are \(\left(n, \left\lfloor \frac{n}{2} \right\rfloor \right)\) and the edges colored 2 are \(\left(n, \left\lfloor \frac{n}{2} \right\rfloor + M\right)\) for \(n = 0, 1, 2, \ldots, 2M - 1\). The colored-digraph \(G(2M)\) is not, in general, contractive. For example, it will follows from Lemma 1 in Section 2 that \(G(6)\) is not a contractive digraph because both \(p_{12}(1)\) and \(p_{12}(2)\) are cycles in \(G(6)\). However, if \(M\) is a power of 2, then \(G(2M)\) is contractive. This will be proved in Example 8 of Section 5. The contractive digraph \(G(4)\) is the rightmost one in Figure 2; digraph \(G(8)\) is the one in Figure 1.
Example 2 (Discrete Cantor Set). Consider the following infinite family \( H(3M) \) for \( M = 1, 2, \ldots \), of 2-colored-digraphs. Let \( V = \{0, 1, 2, \ldots, 3M - 1\} \). The edges colored 1 are \( \left( n, \left\lfloor \frac{n}{3} \right\rfloor \right) \) and the edges colored 2 are \( \left( n, \left\lfloor \frac{n}{3} \right\rfloor + 2M \right) \) for \( n = 0, 1, 2 \ldots 2M - 1 \). That \( H(3M) \) is contractive if \( M \) is a power of 3 will be proved in Example 8 of Section 5.

Example 3 (Discrete Sierpinski Triangle). Let \( M \) be a positive integer. and \( V = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x + y < M\} \). The edges colored 1 of a family \( S(M) \) of 3-colored-digraphs are \( \left( (m, n), \left( \left\lfloor \frac{m}{3} \right\rfloor, \left\lfloor \frac{n}{3} \right\rfloor \right) \right) \); the edges colored 2 are \( \left( (m, n), \left( \left\lfloor \frac{m}{3} \right\rfloor + M, \left\lfloor \frac{n}{3} \right\rfloor \right) \right) \); and the edges colored 3 are \( \left( (m, n), \left( \left\lfloor \frac{m}{3} \right\rfloor, \left\lfloor \frac{n}{3} \right\rfloor + M \right) \right) \). If \( M \) is a power of two, then \( S(M) \) is a contractive digraph.

Example 4 (Colored-digraphs that are not contractive). Cayley graphs \([3]\) are colored-digraphs, the number \( N \) of colors being the cardinality of the generating set of the underlying group of the Cayley graph. Cayley graphs, however, are not in general contractive digraphs because the necessary condition for contractivity provided in Proposition 1 of Section 2 usually fails to hold.

1.2. Splicing Machines. A circular sequence in \([N]\) is a sequence of the form

\[
(j_1, j_2, \ldots, j_n) = (j_2, j_3, \ldots, j_n, j_1) = (j_3, j_4, \ldots, j_n, j_1, j_2) = \cdots = (j_n, j_1, \ldots, j_{n-2}, j_{n-1}),
\]

where \( j_1, \ldots, j_n \in [N] \). Let \( [N]^* \) denote the set of all circular sequences. Two circular sequences

\( C = (j_1, j_2, \ldots, j_n) \) and \( C' = (j'_1, j'_2, \ldots, j'_n) \) are said to be spliced at position \((s, t)\) to obtain the circular sequence

\[
C \bullet C' = C \bullet_{(s, t)} C' = (j_1, j_2, \ldots, j_s, j'_s+1, j'_s+2, \ldots, j'_t, j_{t+1}, j_{t+2}, \ldots, j_n).
\]

Using clockwise orientation, Figure 3 shows a splicing of two circular sequences. It is easy to verify that splicing operations are associative: \( C \bullet C' = C \bullet C' \) and \( C \bullet C' \bullet C'' = C \bullet (C' \bullet C'') \). Therefore, the circular sequence obtained by multiple splicing does not depend on the order of splicing.

![Figure 3](image_url)

The basic idea behind a splicing machine is for the splicing of circular sequences to take place within a digraph. If \( \gamma_x = z, x_1, \ldots, x_s, z \) and \( \gamma_y = z, y_1, \ldots, y_t, z \) are two circuits of a digraph (in terms of their successive vertices) with common vertex \( z \), then the circuit \( \gamma = z, x_1, \ldots, x_s, z, y_1, y_2, \ldots, y_t, z \) is said to be obtained by splicing \( \gamma_x \) and \( \gamma_y \) at vertex \( z \). Let \( G \) be a digraph whose edges are colored in \([N]\). If \( \gamma \) is a circuit of \( G \), then the mapping \( \gamma \mapsto C_\gamma \) assigns to each circuit of \( G \) a circular sequence in \([N]\). When no confusion arises, we may use the terminology “splicing two circuits in \( G \)” and “splicing the corresponding circular sequences in \([N]^*\)” interchangably. Of course, in this context the splicing positions of two circular sequences is restricted to be a vertex of \( G \).

Given a digraph \( G \) whose edges are colored in \([N]\), we may ask: what circular sequences can be obtained by splicing together cycles of \( G \)? Since any circuit in a digraph can be obtained by splicing cycles, this is equivalent to asking about the set

\[
\Gamma_G = \{ C_p : p \text{ is a circuit in } G \}.
\]
of circuit types of $G$. For such a graph $G$ to be a splicing machine, defined formally below, it is required that every circular sequence can be obtained by splicing a set of cycles of $G$.

**Definition 3.** A splicing machine for $[N]$ is a digraph $G$ whose edges are colored in $[N]$ and such that

1. $\Gamma_G = [N]^*$, 
2. for any $C \in [N]^*$, the circuit in the digraph of type $C$ is unique, and 
3. every edge of $G$ appears in some circuit (hence some cycle) of $G$.

The digraph $G$ can have loops and multiple edges, but is not required to be a colored-digraph in the sense of Definition 1. The set

$$C_G = \{C_p : p \text{ is a cycle in } G\}$$

of cycle types is called the set of generators. Condition (3) in the definition is a minimality requirement: if an edge $e$ of $G$ does not appear in a circuit, then the digraph obtained from $G$ by removing $e$ remains a splicing machine.

**Example 5.** As will be proved in Section 3, the colored-digraph $G$ in Figure 4 is a splicing machine. The set of generators is $C_G = \{0, 1, 10, 110, 100, 0011\}$, where 1 stands for red and 0 for black. The cycles in $G$ corresponding to the circular sequences in $C_G$ are denoted $E, A, C, B, D, F$. The cycles $A, B, C, D, E$ are labeled; $F$ is the cycle 1,2,4,3,1 The average length of the generators is $2\frac{1}{3}$. The circular sequence (0, 0, 0, 1, 0, 1) of length 6, for example, is obtained by the splicing $D \bullet_4 E \bullet_3 C$. The corresponding circuit after splicing is 3,2,4,3,2,3. Although not required by Definition 3, the digraph $G$ in this example is a contractive digraph.

![Figure 4. Splicing machine; see Example 5.](image)

1.3. **Organization and Results.** Every contractive digraph $G = (V, E, c)$ has a unique non-trivial strong component, called the attractor of $G$. The attractor is the central object associated with a contractive digraph. The definition of the attractor (Definition 4), the proof of its existence and uniqueness (Theorem 1), and an investigation of its properties is the subject of Section 2. In particular, the attractor is the unique fixed point of a natural operator defined on the set of subsets of $V$ (Theorem 2).

The objective of a splicing machine is to obtain, in an efficient way, from a finite set $C$ of generators, all circular sequences. Given the results of Section 2, it is not hard to show that the attractor of a contractive digraph is a splicing machine. This is the subject of Section 3, in particular Theorem 3. This section also address questions about the efficiency of splicing machines.

There is a natural way to address the vertices of a contractive digraph. This is explained in Section 4, in particular Corollary 2.

Although the paper is completely graph theoretic, one motivation is the Banach fixed point theorem [1] from analysis, namely, a contraction on a complete metric space has a unique fixed point. This connection is explained in Section 5, in particular Theorem 4 and Theorem 5.

Open areas of research related to the notions in this paper appear in Section 6.
2. The Attractor

Recall that \([N]^*\) denotes the set of all finite sequences of colors. If \(C \in [N]^*\), then \(\overline{C} = C C C \cdots\) denotes the infinite concatenation. For \(C \in [N]^*\), the terminal vertex of \(p_C(x_0)\) is denoted \(t_C(x_0)\).

**Lemma 1.** If \(G\) be a contractive digraph and \(p_C(x)\) and \(p_C(y)\) are both circuits in \(G\), then \(x = y\).

**Proof.** Assume by way of contradiction that \(p_C(x)\) and \(p_C(y)\) are both circuits in \(G\). If \(x \neq y\), then \(p_C(x)\) and \(p_C(y)\) are \(C\)-parallel paths, contradicting the definition of contractivity. \(\Box\)

**Proposition 1.** Let \(G\) be a contractive digraph. For every \(C \in [N]^*\) there is a unique circuit in \(G\) of type \(C\). In particular, for every \(j \in [N]\) there is a unique loop in \(G\) colored \(j\).

**Proof.** Uniqueness follows from Lemma 1. Concerning existence, let \(x_n = t_{C^n}(x_0)\) for \(n \geq 1\), where \(x_0 \in V\) and \(C^n = C C \cdots C\) concatenated \(n\) times. Since \(V\) is finite, it must be the case that \(x_i = x_j\) for some \(j > i\). If \(j > i + 1\), then \(x_{i+1} \neq x_i\) and \(p_C(x_i)\) and \(p_C(x_{i+1})\) are \(C\)-parallel, contradicting contractivity. Therefore \(x_{i+1} = x_i\), which implies that \(p_C(x_i)\) is a circuit in \(G\) of type \(C\). \(\Box\)

The set of maximal strongly connected subgraphs of a digraph, each called a strong component, partitions the vertex set \([2]\). Call a strong component trivial if it consists of just one vertex with no loop. (A strong component that is a single vertex with loop(s) is non-trivial.) Given a subset \(X \subseteq V\) of a colored-digraph \(G = (V,E,c)\), by abuse of language, we often do not distinguish between \(X\) and the subgraph of \(G\) induced by \(X\).

**Theorem 1.** Given a contractive digraph \(G = (V,E,c)\), there exists a unique non-trivial strong component \(A\) of \(G\). Moreover

1. \(G\) has no edge \((a,b)\) for which \(a \in A\) and \(b \in V \setminus A\);
2. there is no circuit in \(G\) containing a vertex in \(V \setminus A\); and
3. for every \(C \in [N]^*\) and \(x \in V\), the path \(p_C(x)\) eventually enters and remains in \(A\).

**Proof.** By Proposition 1 there is a strong component that contains a loop colored \(1 \in [N]\) at vertex, say \(a\). Let \(A\) be the strong component containing \(a\). This strong component is non-trivial since it contains a loop.

Assume, by way of contradiction, that \(A\) fails to satisfy property (1). Consider an edge \((a,b)\) for which \(a \in A\) and \(b \in V \setminus A\). There can be no path from \(b\) to a vertex of \(A\); otherwise the maximality of \(A\) is contradicted. Therefore the paths \(p_C(a)\) (repeated loop) and \(p_F(b)\) are parallel, contradicting the assumption that \(G\) is contractive.

We next prove that \(A\) satisfies property (2). If \(C \in [N]^*\) is such that \(p_C(x)\) is a circuit such that \(x \in V \setminus A\), then \(p_C(x)\) and \(p_C(a)\) are parallel for any \(a \in A\), contradicting the contractivity of \(G\).

To prove the uniqueness of \(A\), assume, by way of contradiction, that there is a non-trivial strong component \(B \neq A\). Since \(B\) is strongly connected, there is a circuit in \(B\), contradicting property (2) which was proved in the paragraph above.

Concerning property (3), it follows from property (1) that, once a path enters \(A\), it remains in \(A\). Assume, therefore, that there is a \(C \in [N]^*\) and an \(x \in V \setminus A\) such that \(p_C(x)\) is contained in \(V \setminus A\). Then, for any \(a \in A\), the paths \(p_C(x)\) and \(p_C(a)\) are parallel, contradicting the contractivity of \(G\). \(\Box\)

Properties (1-3) in Theorem 1 motivate the following terminology.

**Definition 4.** Given a contractive digraph \(G = (V,E,c)\), the unique non-trivial strong component \(A\) of \(G\) is called the **attractor** of \(G\).

For each of the colored-digraphs in Figure 2 and the colored-digraph of Example 1, the attractor is the digraph itself.
Example 6 (Discrete Cantor Set and Discrete Sierpinski Triangle Revisited). In Example 2, if the attractor of $H(3^k)$ is denoted denote by $A_k$, then

$$A_k = \{ a \in \{0,1,2,\ldots,3^k-1\} : \text{the base 3 representation of } a \text{ does not contain the digit } 1 \}.$$ 

The attractor is a discrete version of the Cantor set. If these points of $A_k$ are scaled by $1/3^k$ and plotted on the real line, then, as $k \to \infty$, the sets $A_k$ approach (in the Hausdorff metric) the classical Cantor set. This geometric description is by way of motivation; it is not intrinsic to the definition of $A_k$.

A similar situation holds for Example 3. If the points of the attractor $A_k$ of $S(2^k)$ are scaled by $1/2^k$ and plotted in $\mathbb{R}^2$, then, as $k \to \infty$, the attractors $A_k$ approach (in the Hausdorff metric) a Sierpinski triangle.

![Figure 5. The vertices of the attractor of $S(4)$; edges have been omitted.](image)

Corollary 1. Let $G = (V, E, c)$ be a contractive digraph with attractor $A$, and let $C \in [\mathbb{N}]^2$. The unique circuit in $G$ of type $C$, as insured by Proposition 1, is contained in $A$.

Proof. Statement (2) of Theorem 1 implies that there is no circuit that contains a point of $V \setminus A$. □

Let $G = (V, E, c)$ be a colored-digraph. Define a map $T : H(V) \to H(V)$ from the set $H(V)$ of all non-empty subsets of $V$ to itself as follows. For $x \in V$ and $X \subseteq V$, define

$$T(x) = \{ y \in V : (x,y) \in E \} \quad \text{and} \quad T(X) = \{ T(x) : x \in X \}.$$ 

Let $T^n(X) = T \circ T \circ \cdots \circ T(X)$, where it is an $n$-fold composition, and $T^0(X) = X$. The set $T^n(X)$ is the set of vertices of $V$ reachable from a vertex of $X$ by some path of length $n$. Call a subset $X \in H(V)$ a fixed point of $T$ if $T(X) = X$.

Theorem 2. A contractive digraph $G$ with attractor $A$ has the following properties:

1. $A$ is the unique fixed point of $T$, and
2. there is an integer $n_0$ such that $T^n(X) = A$ for every $X \subseteq V$ and every $n \geq n_0$.

Proof. Statement (1): By statement (1) of Theorem 1, there is no edge $(a, b)$ with $a \in A$ and $b \in V \setminus A$. Therefore $T(A) \subseteq A$. Given any two vertices $a_1, a \in A$, strong connectivity implies there is a path from $a_1$ to $a$ in $A$. If $a_2$ is the vertex on this path just before $a$, then $a \in T(a_2)$. Therefore $A \subseteq T(A)$ and hence $T(A) = A$.

Statement (2): If $n_1 = |V \setminus A|$ and $x \in V$, then it follows from Theorem 1 that any path of length $n_1$ must terminate in $A$. Therefore

$$T^n(X) \subseteq A \quad \text{for every } X \subseteq V \text{ and every } n \geq n_1.$$ 

Let $a$ be the vertex of the loop colored $1$, whose existence is insured by Proposition 1. By Corollary 1, vertex $a$ lies in $A$. From Equation (2), the fact that $a \in T(a)$, and by the strong connectivity of $A$, there is an $n_2 \geq n_1$ such that

$$a \in T^n(X) \quad \text{for every } X \subseteq V \text{ and every } n \geq n_2.$$
From $a \in T(a)$, it also follows that $\{a\} \subseteq T(a) \subseteq T^2(a) \subseteq \cdots$. From this and from the strong connectivity of $A$ it follows that there is an $n_0 \geq n_2$ such that

$$T^n(a) = A \quad \text{for all} \quad n \geq n_0.$$  

From Equations (2), (3), and (4) it follows that $T^n(X) = A$ for every $X \subseteq V$ and every $n \geq n_0$.

To prove the uniqueness of the fixed point of $T$, assume that $T(A_1) = A_1$ and $T(A_2) = A_2$. By what was proved above, we have $A_1 = T^{n_0}(A_1) = A = T^{n_0}(A_2) = A_2$. □

3. Splicing Machines

The subject of this section is the relationship between contractive digraphs and splicing machines. If $A$ is the attractor of a contractive digraph, then the digraph $A$ itself is a contractive digraph.

Theorem 3. If $A$ is the attractor of a contractive digraph, then $A$ is a splicing machine.

Proof. Assume that $A$ is the attractor of a cotractive digraph. Let $C$ be any circular sequence. According to Proposition 1, there is a unique circuit in $A$ of type $C$. Therefore, conditions (1) and (2) in Definition 3 hold. By the strong connectivity of $A$, every edge in $A$ lies on a cycle in $A$. Therefore, condition (3) in Definition 3 holds. □

The digraphs in Figures 1 and 4 are both attractors of contractive digraphs; hence both are splicing machines. Using 0 for black and 1 for red, the set of generators of the splicing machine in Figure 4 is $C_1 = \{0, 1, 10, 110, 100, 0011\}$. The set of generators of the splicing machine in Figure 1 is $C_2 = \{0, 1, 10, abed\}$, where $a$ can take the value 0 or 10, $b$ can take the value 0 or 100, $c$ can take the value 1 or 01, and $d$ can take the value 1 or 011; hence $C_2$ has 19 elements. The average length of the generators in $C_2$ is about 6.1 in contrast to about 2.3 for the generators in $C_1$.

Whether the converse of Theorem 3 holds is open. If $G$ is a splicing machine, then it is not hard to show that $G$ must be strongly connected. If $G$ is a colored-digraph in the sense of Definition 1, then it is not hard to show that $G$ must be the attractor of a contractive digraph. So the question remains:

Question. If $G$ is a splicing machine, then does it follow that $G$ must be a colored-digraph?

An algorithmic question also naturally arises.

Question Find an algorithm to determine whether or not a given set $C \subset \hat{[N]}^*$ is the set of generators of a splicing machine.

Factors relevant to the “efficiency” of a splicing machine $G$ include the following:

(1) The number $n$ of vertices in $G$. Note that, if the splicing machine is the attractor of a contractive digraph on $n$ vertices with the number $N$ of colors fixed, then the number of edges is $nN$, linear in $n$.

(2) The number of cycles in $G$.

(3) The number of splices required to obtain a given circular sequence $C$.

(4) The average length of the cycles in $G$. In a splicing machine with a fixed number of vertices, the average number of splices required to obtain a circular sequence is likely inversely proportional to the average length of the cycles in the splicing machine.

Of course, these are conflicting goals. The splicing machine that consists of a single vertex and $N$ loops at this vertex obviously has small order, namely $n = 1$. On the other hand, to obtain a circular sequence of length $k$ with this splicing machine requires $k$ splices, the largest possible. Questions 2 and 3 in Section 6 are pertinent to the above parameters.
4. Address Map

A scheme is developed in this section for addressing the vertices of the attractor of a contractive digraph. An address is an element of $[N]^*$. Basically, a sequence $C \in [N]^*$ is an address of vertex $a$ in the attractor if all paths of type $C$ lead to $a$, independent of the initial vertex.

**Definition 5.** Given a contractive digraph $G$ with attractor $A$ and a vertex $a \in A$, a sequence $C \in [N]^*$ is called an address of $a$ if $t_C(x) = a$ for all $x \in V$.

Let $C$ be an address of $a \in A$ and $C' \in [N]^*$ be an arbitrary finite color sequence. Clearly, the concatenation $CC'$ is an address of $t_{C'}(a)$; and clearly $C'C$ is another address of $a$. This last fact motivates the following definition.

**Definition 6.** Given a contractive digraph $G$ with attractor $A$ and a vertex $a \in A$, a sequence $C = (j_1, j_2, \ldots, j_k) \in [N]^*$ is called a minimal address of $a$ if $t_C(x) = a$ for all $x \in V$, but $t_{j_2,j_3,\ldots,j_k}(x) \neq a$ for some $x \in V$.

Figure 6 shows minimal addresses of the vertices of two attractors. Note that, for the attractor on the right, a minimal address does not have to be unique. We next show that, every vertex in the attractor of a contractive digraph has an address, hence a minimal address.

![Figure 6. Addresses of the vertices of G where black = 1, red = 2.](image)

Let $[N]^k$ denote the set of sequences of length $k$ of elements of $[N]$.

**Proposition 2.** Let $G = (V, E, c)$ be a contractive digraph with attractor $A$. There is an integer $K$ such that for any $m \geq K$ and for any $C \in [N]^m$ we have $t_C(x) = t_C(y)$ for every $x, y \in V$.

**Proof.** Let $k = |A|^2$; let $C = (j_1, j_2, \ldots, j_k)$; and let $x, y \in A$. Consider the sequence of ordered pairs in $A \times A$ defined by

$$(x_0, y_0) = (x, y), \quad (x_1, y_1) = (t_{j_1}(x), t_{j_1}(y)), \quad (x_2, y_2) = (t_{j_1,j_2}(x), t_{j_1,j_2}(y)), \ldots,$$

$$(x_k, y_k) = (t_C(x), t_C(y)).$$

Since $A$ is finite, there must be two terms, say $(x_s, y_s)$ and $(x_t, y_t)$, $s < t$, in this sequence that are identical. If $x_s \neq y_s$, then letting $C' = (j_{s+1}, \ldots, j_t)$, the paths $p_{C'}(x_s)$ and $p_{C'}(y_s)$ are $C$-parallel, contradicting the contractivity of $G$. Therefore $x_s = y_s$, which implies that $t_C(x) = t_C(y)$.

In the paragraph above, $x, y \in A$. Now consider any $x, y \in V$. By statement (2) of Theorem 2, there is an $n$ such that $t_C(x) \in A$ for every $C \in [N]^n$ and every $x \in V$. Therefore, if $C \in [N]^m$ for any $m \geq K = n + k$, then $t_C(x) = t_C(y)$ for all $x, y \in V$. □

**Definition 7.** Given a contractive digraph $G$ with attractor $A$ and constant $K$ as in Proposition 2, let $[N]^{\geq K} = \bigcup_{m \geq K} [N]^m$. Define a map $\pi : [N]^{\geq K} \to A$, called the address map, by

$$\pi(C) = t_C(x),$$

which is independent of $x \in V$ by Proposition 2.

**Corollary 2.** Every vertex in the attractor of contractive digraph has a minimal address.
**Proof.** Since \( T^n(A) = A \) for all \( n \) by Theorem 2, the map \( \pi \) is surjective. Every string in \( \pi^{-1}(a) \) is an address of the vertex \( a \in A \). Since every \( a \in A \) has an address, every such \( a \) has at least one minimal address. \( \square \)

Define an inverse shift map \( s_i : [N]^* \to [N]^* \) by \( s_i(c) = Ci \) for \( i \in [N] \). Given \( K \) as in Proposition 2 and Definition 7, it is routine to check that the following diagram commutes.

\[
\begin{array}{ccc}
[N]^{\geq K} & \xrightarrow{\pi} & [N]^{\geq K} \\
\downarrow & & \downarrow \\
A & \xrightarrow{\pi} & A \\
\end{array}
\]

Interpreting the diagram: appending \( i \in [N] \) to the end of an address of a point \( x \) is an address of a point \( y \) that is adjacent to \( x \) and such that \( c(x, y) = i \).

**Example 7** (Discrete Interval and Discrete Cantor Set Revisited). For the colored-digraphs \( G(2^k), k \geq 0 \), and \( H(3^k), k \geq 0 \), of Examples 1 and 2, denote the respective attractors by \( A_{G,k} \) and \( A_{H,k} \), respectively. Addresses for the vertices of \( A_{G,k} \) and \( A_{H,k} \) are used to show that \( A_{G,k} \) and \( A_{H,k} \) are isomorphic as colored-digraphs. Here isomorphism, denoted by \( \cong \), is a bijection of the vertex sets that preserves directed edges and colors.

The vertices of \( A_{G,k} \) are \( V_{G,k} := \{0,1,2,\ldots,2^k-1\} \), and the vertices of \( A_{H,k} \) are \( V_{H,k} := \{a \in \{0,1,2,\ldots,3^k-1\} : \text{the base 3 representation of } a \text{ does not contain the digit } 1\} \). Let \( a \in A_{G,k} \) and let \( a = \alpha_{k-1}\alpha_{k-2}\cdots\alpha_0 \), the right hand side being the binary representation of \( a \). If \( \beta_i = \alpha_i + 1 \) for \( i \geq 0 \), it is not hard to show that

\[
\pi(\beta_0\beta_1\cdots\beta_{k-1}) = a,
\]

in other words \( \beta_0\beta_1\cdots\beta_{k-1} \) is an address of \( a \).

Likewise, let \( a \in A_{H,k} \) and let \( a = \alpha_{k-1}\alpha_{k-2}\cdots\alpha_0 \), the right hand side being the ternary representation of \( a \). Let \( \beta_i = 1 \) if \( \alpha_i = 2 \) and otherwise \( \beta_i = \alpha_i = 0 \), for all \( i \geq 0 \). It is not hard to show that

\[
\pi(\beta_0\beta_1\cdots\beta_{k-1}) = a,
\]

in other words \( \beta_0\beta_1\cdots\beta_{k-1} \) is an address of \( a \).

Note that \( |V_{G,k}| = |V_{H,k}| = 2^k \), and the set of all address of vertices in \( V_{G,k} \) and in \( V_{H,k} \) are both \([2]^k\). If, in terms of vertex addresses, the map \( \phi : V_{G,k} \to V_{H,k} \) is defined by \( \phi(c) = C \) for all \( C \in [2]^k \), then the commutative diagram (5) implies that \( \phi \) is an isomorphism and

\[
A_{G,k} \cong A_{H,k}
\]

for all \( k \geq 1 \). Although the geometric motivation is quite different for the two families \( G(2^k) \) and \( H(2^k) \) of colored-digraphs, their attractors are isomorphic.

## 5. Banach Fixed Point Theorem

The connection between contractive digraphs and the Banach fixed point theorem is the subject of this section. Let \( d : V \times V \to \mathbb{R} \) be a metric on the space \( V \). A function \( f : V \to V \) is a contraction on the metric space \( (V, d) \) if there is a real number \( 0 \leq r < 1 \) such that \( d(f(x), f(y)) \leq r d(x, y) \) for all \( x, y \in V \). According to the Banach fixed point theorem, a contraction on complete metric space has a unique fixed point.

The situation becomes more interesting when there is more than one contraction on \( (V, d) \). Let \( \mathcal{F} = \{f_1, f_2, \ldots, f_N\} \) for \( N \geq 1 \), be a set of contractions on a metric space \( V \). In the fractal literature, \( \mathcal{F} \) is called an iterated function system. Taking a graph theoretic point of view, define \( G(V, \mathcal{F}) \) to be a digraph with vertex set \( V \) and, for every two points \( x, y \in V \), there is an edge from \( x \) to \( y \) colored \( i \in \{1, 2, \ldots, N\} \) if and only if \( v = f_i(u) \). The digraph \( G(V, \mathcal{F}) \) is a colored-digraph in the sense of Definition 1, except that \( V \) may be infinite. In particular, the definition of contractive digraph carries over to this infinite setting.
Consider the special case of a discrete metric space. A metric space \((V,d)\) is discrete if, for each point \(x \in V\), there is a ball centered at \(x\) containing only \(x\), i.e., if and only if the metric induces the discrete topology on \(V\). The compact subsets a discrete metric space are the finite subsets. A metric space \((V,d)\) is uniformly discrete if there exists an \(\epsilon > 0\) such that for each point \(x \in V\) there is a ball of radius \(\epsilon\) centered at \(x\) containing only \(x\). For a finite space \(V\), discrete and uniformly discrete are equivalent.

**Remark 1.** A standard example of a discrete metric on a set \(V\) is

\[
d_0(x,y) = \begin{cases} 
1 & \text{if } x \neq y \\
0 & \text{if } x = y.
\end{cases}
\]

This metric, however, is not relevant in our context because any contraction on \((V,d_0)\) must be a constant function.

**Theorem 4.** If \((V,d)\) is a uniformly discrete metric space for which each function in \(F = \{f_1, f_2, \ldots, f_N\}\) is a contraction on \(V\) with respect to \(d\), then \(G(V,F)\) is a contractive digraph.

**Proof.** Assume that \(G(V,F)\) is not a contractive digraph. Then there exists \(C = (j_1, j_2, \ldots) \in [N]^\infty\) and \(C\)-parallel paths \(P_C(x_0) = x_0, x_1, x_2, \ldots\) and \(P_C(y_0) = y_0, y_1, y_2, \ldots\) and an \(0 \leq r < 1\) such that

\[
d(x_0, y_0) \geq \frac{1}{r} d(f_{j_1}(x_0), f_{j_1}(y_0)) = \frac{1}{r} d(x_1, y_1) \\
\geq \frac{1}{r^2} d(f_{j_2}(x_1), f_{j_2}(y_1)) = \frac{1}{r^2} d(x_2, y_2) \\
\geq \cdots \geq \frac{1}{r^n} d(x_n, y_n) \geq \ldots,
\]

which implies that the metric space is not uniformly discrete.

Note that a discrete metric space is complete. Let \(F = \{f_1, f_2, \ldots, f_N\}\) be a set of contractions on a discrete metric space \((V,d)\). The unique fixed point of the contraction \(f_i \in F\) is the unique vertex of \(G(V,F)\) on which there is a loop colored \(i\) (as guaranteed by Proposition 1). The operator

\[
T(X) = \bigcup_{f \in F} f(X)
\]

defined on finite non-empty subsets \(X\) of \(V\), often called the Hutchinson operator [4], is exactly the operator \(T : H(X) \to H(V)\) in Equation (1) when \(V\) is finite. By Theorem 2, the unique fixed point of \(T\) is the attractor of \(G(V,F)\).

For \(V\) finite, the converse of Theorem 4 holds. Recall that two colored-digraphs are isomorphic, denoted by \(\simeq\), if there is a bijection of the respective vertex sets that preserves directed edges and colors.

**Theorem 5.** If \(G = (V,E,c)\) is a (finite) contractive digraph, then there exists a discrete metric \(d : V \times V \to \mathbb{R}\) and a set \(F\) of contractions on \(V\) such that \(G(V,F) \simeq G(V,E,c)\).

**Proof.** Given a (finite) contractive digraph \(G = (V,E,c)\), define functions \(f_j : V \to V\) for \(j = 1, 2, \ldots, N\) by setting \(f_j(u) = v\) for each \(u \in V\), where \((u,v)\) is the unique edge colored \(c\). Let \(F = \{f_1, f_2, \ldots, f_N\}\). By construction \(G(V,F) \simeq G(V,E,c)\).

To define the metric for which each \(f \in F\) is a contraction, consider the graph \(G_2\) whose vertex set is \(\binom{V}{2}\) and whose edge set is

\[
\{(x,y) : x \neq y \in V\}.$

Lemma 1 implies that \(G_2\) is acyclic. Therefore the ordering on \(\binom{V}{2}\), defined by \(\{a,b\} \preceq \{c,d\}\) if and only if there exists a (directed) path in \(G_2\) from \(\{c,d\}\) to \(\{a,b\}\), is a partial order. Consider a linear extension

\[
\{x_1,y_1\} \prec^* \{x_2,y_2\} \prec^* \{x_3,y_3\} \prec^* \cdots \prec^* \{x_n,y_n\}
\]
of this partial order, where $n = \binom{V}{2}$. It is clearly possible to define $d : V \times V \to \mathbb{R}$ so that, for all $x, y \in V$,

$$
\begin{align*}
    d(x, x) &= 0 \\
    d(x, y) &= d(y, x) \\
    d(x_1, y_1) &< d(x_2, y_2) < d(x_3, y_3) < \cdots < d(x_n, y_n) \\
    d(x_n, y_n) &< 2d(x_1, y_1).
\end{align*}
$$

This function $d$ is clearly a discrete metric on $V$. To show that each $f \in \mathcal{F}$ is a contraction, let $x, y \in V$. By the definition of the partial order, we have $\{f(x), f(y)\} \sim \{x, y\}$. (Note that it is not possible that $\{f(x), f(y)\} = \{x, y\}$ by Lemma 1.) By the definition of linear extension, we have $\{f(x), f(y)\} \prec^* \{x, y\}$, which implies that $d(f(x), f(y)) < d(x, y)$. Since this is true for every pair $x, y$ and since $V$ is finite, there is an $0 \leq r < 1$ such that $d(f(x), f(y)) < r d(x, y)$ for all $x, y \in V$. 

**Example 8** (The Discrete Interval and the Discrete Cantor Set Revisited). We show that each colored-digraph in the family $G(2^k)$, $k \geq 1$, of Example 1 (the discrete interval) and in the family $H(3^k)$, $k \geq 1$, of Example 2 (the discrete Cantor set) is a contractive digraph.

For $G(2^k)$, define a function $d : V \times V \to \mathbb{R}$ on its vertex set $V = \{0, 1, 2, \ldots, 2^k - 1\}$ as follows. Let $d(n, n) = 0$ for all $n \in V$; let $d(0, 1) = 1$; and for every $n \in V$, $n > 1$, let $d(n - 1, n) = s$, where $s$ is the largest integer such that $n \equiv 0 \mod 2^s$. In general, for $n > m$, set $d(m, n) = d(n, m) = \sum_{i=m+1}^{n} d(i - 1, i)$. It is not hard to check that $d$ is a metric on $V$ for which the functions $f_1$ and $f_2$ defined in Theorem 5 are contractions. By that theorem, $G(2^k)$ is a contractive digraph for $k \geq 1$.

Likewise, for $H(3^k)$, define a function $d : V \times V \to \mathbb{R}$ on its vertex set $V = \{0, 1, 2, \ldots, 3^k - 1\}$ as follows. Let $d(n, n) = 0$ for all $n \in V$; let $d(0, 1) = 1$; and for every $n \in V$, $n > 1$, let $d(n - 1, n) = s$, where $s$ is the largest integer such that $n \equiv 0 \mod 3^s$. In general, for $n > m$, set $d(m, n) = d(n, m) = \sum_{i=m+1}^{n} d(i - 1, i)$. Again, it is not hard to check that $d$ is a metric on $V$ for which the functions $f_1$ and $f_2$ defined in Theorem 5 are contractions. By that theorem, $H(3^k)$ is a contractive digraph.

### 6. Open Problems

In addition to the questions posed in Section 3, the following are open.

**Question 1.** Several infinite families of contractive digraphs are provided in this paper. Find additional methods for their construction.

The next two question are in reference to splicing machine as discussed in Section 3.

**Question 2.** By Theorem 3, the attractor of a contractive digraph $G = G(V, E, c)$ is a splicing machine, and by Theorem 1, all cycles in a contractive digraph are contained in its attractor. Let $g(G)$ denote the number of cycles in $G$. What bounds can be obtained for

$$
g(n, N) := \min g(G),
$$

where the minimum is taken over all contractive digraphs on $n$ vertices and $N$ colors? What can be said about the extremal digraphs, the contractive digraphs that attain this minimum?

**Question 3.** For a contractive digraph $G = G(V, E, c)$, let $h(G, k)$ denote the average number of splicings required to obtain a circular sequence of length $k$. Some sequences, like $111 \cdots 1$ will take $k$ splicings, some may take no splicing. What bounds can be obtained for

$$
h(n, N, k) := \min h(G, k),
$$

where the minimum is taken over all contractive digraphs on $n$ vertices and $N$ colors? What can be said about the extremal cases, the contractive digraphs that attain the minimum?
The average number of splices may be inversely proportional to the average length of the cycles in a splicing machine. So one can ask about bounds on
\[ C(n, N) := \max C(G), \]
where \( C(G) \) is the average length of the cycles in contractive digraph \( G \), and the maximum is taken over all contractive graphs on \( n \) vertices and \( N \) colors? For \( N \) fixed, does \( \lim_{n \to \infty} C(n, N)/n \) exist?

**Question 4.** Let \( G = (V, E, c) \) be a contractive digraph. By Theorem 5, there exists a metric \( d : V \times V \to \mathbb{R} \) on \( V \) and a set \( \mathcal{F} \) of contractions on \( V \) such that \( G(V, \mathcal{F}) \approx G(V, E, c) \). Define the *contractivity* of \( G \) by:
\[ r(G) = \min_d \max \left\{ \frac{d(f(x), f(y))}{d(x, y)} : x, y \in V, f \in \mathcal{F} \right\}, \]
where the minimum is taken over all contractive metrics \( d \) on \( V \). What bounds can be obtained for \( r(G) \)?

**Acknowledgement**

This work was partially supported by a grant from the Simons Foundation (322515 to Andrew Vince).

**References**


Department of Mathematics, University of Florida, USA

E-mail address: avince@ufl.edu