# A Geometric Characterization of Coxeter Matroids 

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#### Abstract

Coxeter matroids, introduced by Gelfand and Serganova, are combinatorial structures associated with any finite Coxeter group and its parabolic subgroup; they include ordinary matroids as a special case. A basic result in the subject is a geometric characterization of Coxeter matroids, first stated by Gelfand and Serganova. This paper presents a self-contained, simple proof of a more general version of this geometric characterization.


Keywords: matroid, Coxeter group, Coxeter matroid, Bruhat order

## 1. Introduction

Matroids were originally introduced by Hassler Whitney in 1935 to abstract certain properties of linear independence, of circuits and bonds in graphs, and of duality. Now they comprise an essential branch of combinatorics [16]. Matroids have a wide variety of applications, ranging from the geometry of Grassmannians [11] to combinatorial optimization [10]. This paper deals with a generalization of the notion of matroid introduced by Gelfand and Serganova about 1987 [12, 13]. To each finite Coxeter group $W$ and parabolic subgroup $P$, they associated a collection of subsets of $W / P$ called Coxeter matroids (originally called ( $W, P$ )-matroids). Ordinary matroids correspond to the case where $W$ is the symmetric group $S_{n+1}$, and $P$ is its maximal parabolic subgroup. Coxeter matroids and their further generalizations were studied in [1-7, 15].

The original definition of a Coxeter matroid was given in terms of the Bruhat order on $W / P$ (see Section 4). A fundamental result in the theory of Coxeter matroids gives an equivalent geometric definition in terms of the matroid polytope associated with $(W, P)$. This definition (with a sketch of a proof that two definitions are equivalent) appeared in [13]; a complete proof of the equivalence of two definitions for Weyl groups was
given in [15]. In this paper, we extend the result and proof of [15] from Weyl groups to arbitrary finite Coxeter groups. In the process, we also provide a self-contained brief introduction to Coxeter matroids.

In [15], the geometric interpretation of Coxeter matroids corresponding to Weyl groups was used to obtain their characterization in terms of greedy algorithms for a generalized assignment problem. This result generalizes the classical Rado-Edmonds theorem for ordinary matroids. In a forthcoming publication, we will extend this result to arbitrary finite Coxeter groups.

This paper is organized as follows. Definitions concerning Coxeter groups and complexes appear in Section 2. Results on Bruhat order appear in Section 3. The Bruhat order is defined on the collection of cosets $W / P$. Moreover, with respect to any $w \in W$, there is a twisted version of the Bruhat order called the $w$-Bruhat order. Necessary geometric conditions for two elements to be comparable to each other in the $w$-Bruhat order with respect to a single element $w$ of the Coxeter group are known. Theorem 3.1 gives a geometric condition necessary and sufficient for two elements of a Coxeter group to be comparable to each other in the $w$-Bruhat order relative to every element $w$ of the Coxeter group. Coxeter matroids are introduced in Section 4. In Section 5, we introduce matroid Coxeter polytopes and prove the geometric characterization theorem (Theorem 5.2). The proof is based on a general result about polytopes established in [15, Theorem 3.5]. For the convenience of the reader, we reproduce this result in Theorem 5.1.

## 2. Coxeter Groups and Coxeter Complexes

In this section, we collect definitions and notation related to finite Coxeter groups (the standard reference is [8]). Let ( $W, S$ ) be a finite Coxeter system of rank $n$. This means that $W$ is a finite group with the set $S$ consisting of $n$ generators, and with the presentation

$$
<s \in S \mid\left(s s^{\prime}\right)^{m_{s s^{\prime}}}=1>
$$

where $m_{s s^{\prime}}$ is the order of $s s^{\prime}$, and $m_{s s}=1$ (hence, each generator is an involution). The group $W$ is called a Coxeter group. The diagram of $(W, S)$ is the graph where each generator is represented by a node, and nodes $s$ and $s^{\prime}$ are joined by an edge labeled $m_{s s^{\prime}}$ whenever $m_{s s^{\prime}} \geq 3$. By convention, the label is omitted if $m_{s s^{\prime}}=3$. A Coxeter system is irreducible if its diagram is a connected graph. A reducible Coxeter group is the direct product of the Coxeter groups corresponding to the connected components of its diagram. Finite irreducible Coxeter groups have been completely classified and are usually denoted by $A_{n}(n \geq 1), B_{n}(n \geq 2), D_{n}(n \geq 4), E_{6}, E_{7}, E_{8}, F_{4}, G_{2}, H_{3}, H_{4}$, and $I_{2}(m)(m \geq 5, m \neq 6)$, the subscript denoting the rank.

A reflection in $W$ is a conjugate of some element of $S$. Let $T=T(W)$ denote the set of all reflections in $W$. Every finite Coxeter group $W$ can be realized as a reflection group in some Euclidean space $\mathbb{E}$ of dimension equal to the rank of $W$. In this realization, each element of $T$ corresponds to the orthogonal reflection through a hyperplane in $\mathbb{E}$ containing the origin. Each of the irreducible Coxeter groups listed above, except $D_{n}, E_{6}, E_{7}$, and $E_{8}$, is the symmetry group of a regular convex polytope. The group $A_{n}$
is isomorphic to the symmetric group $S_{n+1}$, the set $S$ consisting of the adjacent transpositons $s_{i}=(i, i+1)$.

For a finite Coxeter system ( $W, S$ ), let $\Sigma$ denote the set of all reflecting hyperplanes in $\mathbb{E}$. Let $E^{\prime}=\mathbb{E} \backslash \bigcup_{H \in \Sigma} H$. The connected components of $E^{\prime}$ are called chambers. For any chamber $\Gamma$, its closure $\bar{\Gamma}$ is a simplicial cone in $\mathbb{E}$. These cones and all their faces form a fan in $\mathbb{E}$ called the Coxeter complex and denoted $\Delta:=\Delta(W, S)$.

It is known that $W$ acts simply transitively on the set of chambers. To identify the elements of $W$ with chambers, we choose a fundamental chamber $\Gamma_{0}$ whose facets (i.e., faces of codimension one) are reflecting hyperplanes for the simple reflections $s \in S$, then the bijective correspondence between $W$ and the set of chambers is given by $w \mapsto w\left(\Gamma_{0}\right)$. Two chambers $u\left(\Gamma_{0}\right)$ and $v\left(\Gamma_{0}\right)$ share a facet if and only if $v=u s$ for some $s \in S$. Thus, the Cayley graph $G(W, S)$ of $(W, S)$ can be identified with the dual graph of the complex $\Delta(W, S)$.

A (standard) parabolic subgroup of $W$ is a subgroup generated by some subset of $S$. If $P$ is a parabolic subgroup, we denote by $\Gamma_{0}(P)$ the set of points in $\bar{\Gamma}_{0}$ whose stabilizer in $W$ is exactly $P$. The closure $\overline{\Gamma_{0}(P)}$ is a face of the cone $\overline{\Gamma_{0}}$, and the correspondence $P \mapsto \overline{\Gamma_{0}(P)}$ is a bijection between the set of parabolic subgroups of $W$ and the set of faces of $\overline{\Gamma_{0}}$. The action of $W$ extends this correspondence to a bijection $w P \mapsto w\left(\overline{\Gamma_{0}(P)}\right)$ between the union of left coset spaces $U W / P$ modulo all parabolic subgroups, and the set of all faces of the Coxeter complex. Under this correspondence, the codimension of the face $w\left(\overline{\Gamma_{0}(P)}\right)$ is equal to the number of simple reflections that generate $P$.

## 3. Bruhat Order

In this section we review the main properties of the Bruhat partial order on a Coxeter group $W$; for the proofs, see $[6,8,9]$ or [14]. We will use the notation $u \succeq w$ for the Bruhat order on $W$. It will be defined in two equivalent ways.

For $w \in W$, a factorization $w=s_{1} s_{2} \ldots s_{k}$ into the product of simple reflections is called reduced if it is the shortest possible. Let $l(w)$ denote the length $k$ of a reduced factorization of $w$.

Definition 1. Define $u \succeq v$ if there exists a sequence $v=u_{0}, u_{1}, \ldots, u_{m}=u$ such that $u_{i}=t_{i} u_{i-1}$ for some reflection $t_{i} \in T(W)$, and $l\left(u_{i}\right)>l\left(u_{i-1}\right)$ for $i=1,2 \ldots, m$.

Definition 2. If $u=s_{1} s_{2} \cdots s_{k}$ is a reduced factorization, then $u \succeq v$ if and only if there exist indices $1 \leq i_{1}<\cdots<i_{j} \leq k$ such that $v=s_{i_{1}} \cdots s_{i_{j}}$.

The Bruhat order can be also defined on the left coset space $W / P$ for any parabolic subgroup $P$ of $G$; again there are several equivalent definitions.

Definition 3. Define Bruhat order on $W / P$ by $\bar{u} \succeq \bar{v}$ if there exists a $u \in \bar{u}$ and $v \in \bar{v}$ such that $u \succeq v$.

It is known (see, e.g., [14]) that any $\operatorname{coset} \bar{u} \in W / P$ has a unique representative of minimal length. We denote this representative by $\bar{u}_{\text {min }}$.

Definition 4. We have $\bar{u} \succeq \bar{v}$ in the Bruhat order on $W / P$ if and only if $\bar{u}_{\min } \succeq \bar{v}_{\min }$.

As in Section 2, we represent $W$ as a reflection group in some Euclidean space $\mathbb{E}$; we retain all the notation in Section 2 . Fix any point $\delta \in \Gamma_{0}(P)$. Since, by definition, the stabilizer of $\delta$ in $W$ is $P$, we can unambiguously define the point $\bar{u} \delta \in \mathbb{E}$ for any $\bar{u} \in W / P$. Using Definitions 1 and 3, we obtain the following geometric definition of the Bruhat order on $W / P$.

Definition 5. We have $\bar{u} \succeq \bar{v}$ in the Bruhat order on $W / P$ if there exists a sequence of points $\bar{v} \delta=\delta_{0}, \delta_{1}, \ldots, \delta_{m}=\bar{u} \delta$ in $\mathbb{E}$ such that $\delta_{i}=t_{i} \delta_{i-1}$ for some reflection $t_{i} \in T(W)$, and, for all $i$, the reflecting hyperplane of $t_{i}$ separates $\delta_{i}$ from the chamber $\Gamma_{0}$.

This definition admits a convenient reformulation in terms of the standard inner product $(\cdot, \cdot)$ on $\mathbb{E}$.

Definition 6. We have $\bar{u} \succeq \bar{v}$ in the Bruhat order on $W / P$ if there exists a sequence of points $\bar{v} \delta=\delta_{0}, \delta_{1}, \ldots, \delta_{m}=\bar{u} \delta$ in $\mathbb{E}$ such that $\delta_{i}=t_{i} \delta_{i-1}$ for some reflection $t_{i} \in T(W)$, and $\left(\delta_{i}, \eta\right)<\left(\delta_{i-1}, \eta\right)$ for all $i$ and all $\eta \in \Gamma_{0}$.

To illustrate the above definitions, consider the example of type $A_{n}$, where $W=S_{n+1}$ is the group of all permutations of $[1, n+1]=\{1,2, \ldots, n+1\}$. If $A=\left\{a_{1}<\cdots<a_{k}\right\}$ and $B=\left\{b_{1}<\cdots<b_{k}\right\}$ are two subsets of the same cardinality in $[1, n+1]$ arranged in increasing order, we denote by $A \geq B$ the partial order given by

$$
a_{1} \geq b_{1}, a_{2} \geq b_{2}, \ldots, a_{k} \geq b_{k}
$$

This partial order is known as the Gale order [10]. It is well known that the Bruhat order on $S_{n+1}$ can be described in terms of the Gale order as follows: $u \succeq v$ if and only if, for any $k=1, \ldots, n$, the subsets $u([1, k])$ and $v([1, k])$ satisfy $u([1, k]) \geq v([1, k])$. The Bruhat order on $W / P$ looks especially simple when $P$ is a maximal parabolic subgroup in $W$ generated by $s_{1}, \ldots, s_{k-1}, s_{k+1}, \ldots, s_{n}$, for some $k=1, \ldots, n$. This subgroup is naturally isomorphic to $S_{k} \times S_{n+1-k}$, and the correspondence $u \mapsto u([1, k])$ is a bijection of $S_{n+1} /\left(S_{k} \times S_{n+1-k}\right)$ with the set of all $k$-element subsets of $[1, n+1]$. Under this correspondence, the Bruhat order on $S_{n+1} /\left(S_{k} \times S_{n+1-k}\right)$ corresponds to the Gale order on $k$-element subsets.

Returning to the general case, we note that, in view of Definition 6 , if $\bar{u} \succ \bar{v}$ in the Bruhat order on $W / P$, then $(\bar{u} \delta, \eta)<(\bar{v} \delta, \eta)$ for all $\eta \in \Gamma_{0}$. In some special cases this geometric condition is equivalent to $\bar{u} \succ \bar{v}$, for example, this is true in the case $W / P=S_{n+1} /\left(S_{k} \times S_{n+1-k}\right)$ just considered. In general, however, the Bruhat order is not characterized by this geometric condition. We illustrate this by the following counterexample.

Consider the group $W=B_{3}$ with Coxeter presentation

$$
W=\left\langle s_{1}, s_{2}, s_{3} \mid s_{1}^{2}=s_{2}^{2}=s_{3}^{2}=\left(s_{1} s_{2}\right)^{4}=\left(s_{1} s_{3}\right)^{2}=\left(s_{2} s_{3}\right)^{3}=1\right\rangle .
$$

Its geometric realization is the symmetry group of a cube in the standard Euclidean space $\mathbb{R}^{3}$, where the generating reflections are given by

$$
s_{1}=\left(\begin{array}{ccc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right), \quad s_{2}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{array}\right), \quad s_{3}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{array}\right) .
$$

The fundamental chamber can be chosen as follows:

$$
\Gamma_{0}=\left\{(x, y, z) \in \mathbb{R}^{3}: 0<x<y<z\right\} .
$$

Let $P=\{e\}$, and take $u=s_{3} s_{1} s_{2} s_{1} s_{3}$ and $v=s_{2} s_{1} s_{2}$. These two factorizations are easily seen to be reduced. By Definition 2, $u$ and $v$ are incomparable in the Bruhat order on $W$. On the other hand, choosing $\delta=(1,2,3)$, we see that $u \delta=(-3,2,-1)$ and $v \delta=(1,-2,3)$, hence, for any $\eta=(x, y, z) \in \Gamma_{0}$, we have $(v \delta, \eta)-(u \delta, \eta)=4 x+4(z-$ $y)>0$.

For a general finite Coxeter group $W$ and its parabolic subgroup $P$, we will associate with each $w \in W$ a twisted version of the Bruhat order on $W / P$, which will be called the $w$-Bruhat order and denoted by $\succeq_{w}$.

Definition 7. Define $\bar{u} \succeq_{w} \bar{v}$ in the $w$-Bruhat order on $W / P$ if $w^{-1} \bar{u} \succeq w^{-1} \bar{v}$.
Definitions 5 and 6 have obvious analogues for the $w$-Bruhat order, with $\Gamma_{0}$ replaced by the chamber $w \Gamma_{0}$. Either of these definitions implies at once that, if $\bar{v}=t \bar{u}$ for some reflection $t \in T(W)$, then $\bar{u}$ and $\bar{v}$ are comparable in every $w$-Bruhat order on $W / P$. The converse statement is also true.

Theorem 3.1. Two distinct elements $\bar{u}$ and $\bar{v}$ in $W / P$ are comparable in the w-Bruhat order for every $w \in W$ if and only if $\bar{v}=t \bar{u}$ for some reflection $t \in T(W)$.

Proof. We will deduce the theorem from the following lemma.
Lemma 3.2. For every two elements $\bar{u}$ and $\bar{v}$ in $W / P$, there exists $w \in W$ such that $0 \leq l\left(\left(w^{-1} \bar{v}\right)_{\text {min }}\right)-l\left(\left(w^{-1} \bar{u}\right)_{\text {min }}\right) \leq 1$.

Suppose two elements $\bar{u}$ and $\bar{v}$ of $W / P$ are comparable in every $w$-Bruhat order. Choose $w$ as in Lemma 3.2, and set $\left(w^{-1} \bar{v}\right)_{\text {min }}=w^{-1} v$ and $\left(w^{-1} \bar{u}\right)_{\text {min }}=w^{-1} u$. By Definition 4, we have $w^{-1} v \succeq w^{-1} u$. Now consider two cases. If $l\left(w^{-1} v\right)=l\left(w^{-1} u\right)$, then we obviously have $w^{-1} v=w^{-1} u$, i.e., $v=u$ and hence $\bar{u}=\bar{v}$. If $l\left(w^{-1} v\right)=l\left(w^{-1} u\right)+1$ then, by Definition 1, we have $w^{-1} v=t^{\prime} w^{-1} u$ for some $t^{\prime} \in T(W)$. In this case $v=t u$, and hence $\bar{v}=t \bar{u}$, where $t=w t^{\prime} w^{-1}$. To complete the proof of the theorem, it only remains to notice that $t \in T(W)$ since $T(W)$ is closed under conjugation.

Proof of Lemma 3.2. Choose representatives $u \in \bar{u}$ and $v \in \bar{v}$ so that $i\left(u^{-1} v\right)$ is minimal (that is, $u$ and $v$ are at the minimal distance in the Cayley graph $G(W, S)$ ). Choose a reduced factorization $u^{-1} v=s_{1} s_{2} \ldots s_{m}$. Let $k=[m / 2]$ and $w=u s_{1} s_{2} \ldots s_{k}$. We have $w^{-1} u=s_{k} \ldots s_{i}$ and $w^{-1} v=s_{k+1} \ldots s_{m}$, so $0 \leq l\left(w^{-1} v\right)-l\left(w^{-1} u\right)=m-2 k \leq 1$. It remains to show that $w^{-1} v$ and $w^{-1} u$ are minimal elements in their cosets $w^{-1} \bar{v}$ and $w^{-1} \bar{u}$. Let $w^{-1} v^{\prime}=\left(w^{-1} \bar{v}\right)_{\text {min }}$ and $w^{-1} u^{\prime}=\left(w^{-1} \bar{u}\right)_{\text {min }}$. Assuming that $u^{\prime} \neq u$ or $v^{\prime} \neq v$, we would have
$l\left(u^{\prime-1} v^{\prime}\right)=l\left(\left(w^{-1} u^{\prime}\right)^{-1}\left(w^{-1} v^{\prime}\right)\right) \leq l\left(w^{-1} u^{\prime}\right)+l\left(w^{-1} v^{\prime}\right)<l\left(w^{-1} u\right)+l\left(w^{-1} v\right)=l\left(u^{-1} v\right)$.
which contradicts the choice of $u$ and $v$. This contradiction completes the proof of Lemma 3.2.

## 4. Coxeter Matroids

Following [12,13], we now define Coxeter matroids, the central object of study in this paper. Let $(W, S)$ be a finite Coxeter system and $P$ a parabolic subgroup in $W$. A subset $M \subseteq W / P$ is called a Coxeter matroid (for $W$ and $P$ ) if, for each $w \in W$, there is a unique minimal element in $M$ with respect to the $w$-Bruhat order.

To show that this definition includes ordinary matroids as a special case, recall that one of the many equivalent definitions of ordinary matroids is in terms of the Gale order on $k$-element subsets of a linearly ordered finite set $I$. This order is defined as follows: if $A=\left\{a_{1}, \ldots, a_{k}\right\}$ and $B=\left\{b_{1}, \ldots, b_{k}\right\}$, each with elements arranged in increasing order, then $A \geq B$ if $a_{i} \geq b_{i}$ for $i=1, \ldots, k$. According to [10], a matroid (of rank $k$ ) on a finite set $l$ can be defined as a collection $\mathcal{B}$ of $k$-element subsets of $I$, called bases, such that, for any linear ordering of $I$, there exists a unique base $B \in \mathcal{B}$ minimal in the corresponding Gale ordering. Now if we take $I=[1, n+1]$, then, as shown in the previous section, the set of all $k$-element subsets of $I$ is identified with $W / P$, where $W=S_{n+1}$ and $P=S_{k} \times S_{n+1-k}$. Furthermore, linear orderings of $I$ correspond to permutations $w \in S_{n+1}$. Under this correspondence, the Gale order corresponds to the $w$-Bruhat order on $W / P$. In other words, matroids of rank $k$ on $[1, n+1]$ are precisely Coxeter matroids for $W=S_{n+1}$ and $P=S_{k} \times S_{n+1-k}$.

The next result shows that the concept of a Coxeter matroid for $W$ and an arbitrary parabolic subgroup $P$ can be reduced to the concept of Coxeter matroid for $W$ and the trivial subgroup $P_{0}=\{e\}$.

Proposition 4.1. Let $P$ be a parabolic subgroup of $W$, and $p: W \rightarrow W / P$ be the natural projection. Then
(1) If $M \subseteq W$ is a Coxeter matroid in $W$, then $p(M)$ is a Coxeter matroid in $W / P$.
(2) If $M \subseteq W / P$ is a Coxeter matroid in $W / P$, then $p^{-1}(M)$ is a Coxeter matroid in $W$.

Proof. Part (1) follows from Definition 3, and part (2) follows from Definition 4.
In particular, for type $A_{n}$, it suffices to study Coxeter matroids in $W=S_{n+1}$. This notion also admits a reformulation in terms of ordinary matroids. To present it we need to recall some standard terminology from matroid theory. Let $\mathcal{B}$ be the collection of bases of a matroid on a finite set $I$. The $\operatorname{rank} \operatorname{rk}(J)$ of a subset $J \subseteq I$ is the maximal cardinality of the intersection $J \cap B$ for $B \in \mathcal{B}$. A subset $F \subseteq I$ is closed with respect to $\mathcal{B}$ if $\mathrm{rk}(F \cup\{x\})=\operatorname{rk}(F)+1$ for all $x \in I-F$. The following proposition was proved in [13].

Proposition 4.2. Let $M \subseteq S_{n+1}$, and for $k=1, \ldots, n$, let $\mathcal{B}_{k}$ be the collection of subsets $w([1, k])$ for $w \in M$. Then $M$ is a Coxeter matroid if and only if each $\mathcal{B}_{k}$ is the collection of bases of a matroid of rank $k$ on $[1, n+1]$, and every subset closed with respect to $\mathcal{B}_{k}$ is also closed with respect to $\mathcal{B}_{k+1}$.

## 5. Geometric Characterization of Coxeter Matroids

To every Coxeter matroid will be associated a convex polytope called the matroid polytope. In the case of an ordinary matroid $M$ on a finite set $I$ with the collection $\mathcal{B}$ of
bases, let $\mathbb{E}$ be a Euclidean space with a given basis $\left(\varepsilon_{i}\right)(i \in I)$. With every $B \in \mathcal{B}$, we associate the vector

$$
\delta_{B}=\sum_{i \in B} \varepsilon_{i} .
$$

The matroid polytope $\Delta(M)$ is the convex hull of the $\delta_{B}$ for all $B \in \mathcal{B}$.
To extend this definition to general Coxeter matroids, let $M \subseteq W / P$, where $W$ is a finite Coxeter group and $P$ is a parabolic subgroup in $W$. The notation that follows was introduced in previous sections. For any $\delta \in \Gamma_{0}(P)$, let $\Delta_{\delta}(M)$ be the convex hull of the points $\bar{w} \delta$ for $\bar{w} \in M$. Note that, since $W$ acts on $\mathbb{E}$ by orthogonal transformations, all the points $\bar{w} \delta$ lie at the same distance from the origin. It follows that all these points are vertices of $\Delta_{\delta}(M)$. It is shown in [7] that the combinatorial type of $\Delta_{\delta}(M)$ does not depend on the choice of $\delta \in \Gamma_{0}(P)$. Moreover, if $\delta$ and $\delta^{\prime}$ are two points in $\Gamma_{0}(P)$, then corresponding edges of $\Delta_{\delta}(M)$ and $\Delta_{\delta^{\prime}}(M)$ are parallel. In view of these results, we will sometimes write $\Delta(M)$ for $\Delta_{\delta}(M)$. If $M$ is a Coxeter matroid, the polytope $\Delta(M)$ is called the matroid polytope of $M$.


Figure 1: Example of a matroid polytope.
Figure 1 shows the matroid polytope of the ordinary rank 2 matroid $M$ on $\{1,2,3,4\}$ with the collection of bases $\mathcal{B}=\{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$. The Coxeter group $W=$ $S_{4}$ is the symmetry group of the regular tetrahedron. The point $\delta$ is taken as the midpoint of an edge of the tetrahedron. The matroid polytope is the square, whose four vertices correspond to the four bases of the matroid.

Before formulating and proving our main result about Coxeter matroids (Theorem 5.2), we establish a general result about convex polytopes in Euclidean spaces. Let $\mathbb{E}$ be a finite-dimensional Euclidean space with inner product $(\cdot, \cdot)$. Let $H_{1}, \ldots, H_{N}$ be a collection of codimension 1 subspaces of $\mathbb{E}$, and set $E^{\prime}=\mathbb{E} \backslash \cup H_{i}$. A vector $\eta \in \mathbb{E}$ is regular if $\eta \in E^{\prime}$. Connected components of $E^{\prime}$ will be called chambers. We associate with every chamber $\Gamma$ a partial ordering $\leq_{r}$ on $\mathbb{E}$ by setting

$$
\xi_{1} \leq \Gamma \xi_{2} \text { if }\left(\xi_{1}, \eta\right) \leq\left(\xi_{2}, \eta\right) \text { for every } \eta \in \Gamma
$$

Theorem 5.1. Let $\Delta$ be a convex polytope in $\mathbb{E}$. The following conditions are equivalent.
(1) Every edge of $\Delta$ is orthogonal to some hyperplane $H_{i}$.
(2) For every regular vector $\eta$, the maximum $\max _{\xi \in \Delta}(\xi, \eta)$ is attained at a unique vertex of $\Delta$.
(3) For every chamber $\Gamma$, there is a unique vertex $\xi(\Gamma)$ of $\Delta$ maximal with respect to the ordering $\leq r$.
Proof. (3) $\Rightarrow$ (2). If $\eta \in \Gamma$, then, by definition, $\max _{\xi \in \Delta}(\xi, \eta)$ is attained at the unique vertex $\xi(\Gamma)$.
(2) $\Rightarrow$ (3). For each regular $\eta$, denote by $\xi(\eta)$ the unique vertex in $\Delta$ at which $\max _{\xi \in \Delta}(\xi, \eta)$ is attained. The map $\eta \mapsto \xi(\eta)$ is locally constant, and hence, constant on each chamber $\Gamma$. The value of this map at points of $\Gamma$ is, by definition, the vertex $\xi(\Gamma)$.
$(2) \Rightarrow(1)$. Assume that an edge $[\alpha, \beta]$ of $\Delta$ is not orthogonal to any $H_{i}$. This implies that there exists a regular vector $\eta_{0}$ such that $\left(\alpha, \eta_{0}\right)=\left(\beta, \eta_{0}\right)$. Now let $\eta \in \mathbb{E}$ be a vector such that $\max _{\xi \in \Delta}(\xi, \eta)$ is attained along the whole edge $[\alpha, \beta]$. Adding, if necessary, a vector $\varepsilon \eta_{0}$ with sufficiently small $\varepsilon>0$ to $\eta$ results in a regular vector $\eta^{\prime}=\eta+\varepsilon \eta_{0}$. But $\left(\alpha, \eta^{\prime}\right)=\left(\beta, \eta^{\prime}\right)$, which contradicts (2).
(1) $\Rightarrow(2)$. Let $\eta$ be regular and $\xi_{0}$ a vertex at which $\max _{\xi \in \Delta}(\xi, \eta)$ is attained. Since $\eta$ is regular, it is not orthogonal to any edge $\left[\xi_{0}, \xi_{1}\right]$ of $\Delta$, which implies that $\left(\xi_{0}, \eta\right) \neq\left(\xi_{1}, \eta\right)$. Hence, $\left(\xi_{0}, \eta\right)>\left(\xi_{1}, \eta\right)$ for all $\xi_{1}$ adjacent to $\xi_{0}$, and therefore, also for all $\xi_{1} \in \Delta, \xi_{1} \neq \xi_{0}$.

Now everything is ready for the proof of our main result.
Theorem 5.2. Let $(W, S)$ be a finite Coxeter system, $P$ a parabolic subgroup of $W$, and $\delta$ any point in $\Gamma_{0}(P)$. Let $M$ be a subset of $W / P$ and $\Delta(M)=\Delta_{\delta}(M)$ the corresponding polytope. The following statements are equivalent.
(1) $M$ is a Coxeter matroid.
(2) Every edge of $\Delta(M)$ is orthogonal to one of the reflecting hyperplanes.
(3) For any two adjacent vertices $\alpha$ and $\beta$ of $\Delta(M)$, there is a reflection $t \in W$ such that $t \alpha=\beta$.
(4) For any regular $\eta \in \mathbb{E}$, the linear functional $\xi \mapsto(\xi, \eta)$ attains its maximum on $\Delta(M)$ at a unique point.
(5) For any chamber $\Gamma$, the polytope $\Delta(M)$ has a unique maximal point with respect to the ordering $\leq r$.
Proof. The equivalence of (2), (4), and (5) follows from Theorem 5.1 applied to the collection of reflecting hyperplanes. The implication (1) $\Rightarrow(5)$ is an easy consequence of Definition 6 of the Bruhat order on $W / P$ (see the discussion in Section 3). To complete the proof, it suffices to show that $(2) \Rightarrow(3)$ and $(3) \Rightarrow(1)$.
(2) $\Rightarrow$ (3). Let $[\alpha, \beta]$ be an edge of $\Delta(M)$ and $t \in W$ the reflection in the hyperplane orthogonal to $[\alpha, \beta]$. Then $\alpha, \beta$, and $t \alpha$ lie on a line. Since all three points are at the same distance from the origin, it follows that $\beta=t \alpha$.
(3) $\Rightarrow$ (1). Let $w$ be any element of $W$ and $\Gamma=w \Gamma_{0}$ the corresponding chamber. Choose a point $\eta$ in $\Gamma$ and let $\bar{v}$ be any element of $M$ that maximizes $(\bar{v} \delta, \eta)$. We claim that $\bar{u} \succeq_{w} \bar{v}$ for any $\bar{u} \in M$, hence, $M$ is a Coxeter matroid. Since $(\bar{u} \delta, \eta) \leq(\bar{\delta} \delta, \eta)$, there is a sequence $\bar{u}=\bar{u}_{0}, \bar{u}_{1}, \ldots, \bar{u}_{m}=\bar{v}$ in $M$ such that each $\left[\bar{u}_{i-1} \delta, \bar{u}_{i} \delta\right]$ is an edge of $\Delta(M)$ and $\left(\bar{u}_{i-1} \delta, \eta\right)<\left(\bar{u}_{i} \delta, \eta\right)$ for $i=1, \ldots, m$. Combining (3) with Theorem 3.1, we see that any two elements $\bar{u}_{i-1}$ and $\bar{u}_{i}$ are comparable in the $w$-Bruhat order. Moreover, by Definition 6, $\bar{u}_{i-1} \succ_{w} \bar{u}_{i}$, hence, $\bar{u}=\bar{u}_{0} \succ_{w} \bar{u}_{1} \succ_{w} \cdots \succ_{w} \bar{u}_{m}=\bar{v}$, as claimed.

We illustrate the theorem in Figure 1. Condition (2) of Theorem 5.2 requires that every edge of $\Delta(M)$ be orthogonal to a reflecting hyperplane, i.e., be parallel to an edge of the tetrahedron. Since this is the case for the square in the figure, the collection $\mathcal{B}=\{\{1,3\},\{1,4\},\{2,3\},\{2,4\}\}$ is indeed the collection of bases of a matroid.

We conclude with the following remark: in view of Theorem 5.2, if any of the statements (2), (3), (4) or (5) is true for some choice of $\delta$, then they are all true for all choices of $\delta$.

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