Cycles in a Graph Whose Lengths Differ by One or Two

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This note is dedicated to the memory of our friend and colleague Paul Erdős.

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Abstract: Several problems concerning the distribution of cycle lengths in a graph have been proposed by P. Erdös and colleagues. In this note two variations of the following such question are answered. In a simple graph where every vertex has degree at least three, must there exist two cycles whose lengths differ by one or two? © 1998 John Wiley & Sons, Inc. J Graph Theory **27:** 11–15, 1998

1. INTRODUCTION

Several questions concerning the distribution of cycle lengths in a graph have been posed by P. Erdős and colleagues. One of these questions is the following.

Question. In a simple graph where every vertex has degree at least three, must there exist two cycles whose lengths differ by one or two?

Other open problems concerning the existence of cycles of specified length are mentioned by Erdős in [2]. For example, Erdős and A. Gyárfás asked: If G is a graph with minimum degree three, must G have a cycle of length 2^r for some integer r? More generally, Erdős asked (and

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offered \$100) for a proof of the existence or nonexistence of a sequence $a_1 < a_2 < \cdots$ of density 0 and an absolute constant c for which every graph on n vertices and cn edges contains a cycle of length a_i for some i.

In this note, the above Question is answered in the affirmative. The conclusion cannot be strengthened to 'two cycles whose lengths differ by one' because, in a bipartite graph, two such cycles do not exist. However, the assertion can be pushed in two directions, given in the following two theorems. Let K_n, C_n, P_n denote the complete graph, cycle and path, respectively, on n vertices, and let $K_{m,n}$ denote the complete bipartite graph with parts of sizes m and n.

Theorem 1. *With the exception of K_1 and K_2, every simple graph having at most two vertices of degree less than three contains two cycles whose lengths differ by one or two.*

Theorem 2. *Every nonbipartite* 3*-connected graph has two cycles whose lengths differ by one.*

Theorem 1 is best possible in the sense that each of the graphs C_3, P_3 and $K_{2,3}$ have three vertices of degree less than three but contain no two cycles whose lengths differ by one or two. However, if one allows additional exceptions, then Theorem 1 can be extended. With exactly twelve exceptions, every simple graph having at most *three* vertices of degree less than three contains two cycles whose lengths differ by one or two. In addition to K_1 and K_2, the exceptions are C_3, P_3 and $K_{2,3}$, and the seven graphs obtained from C_3, P_3 and $K_{2,3}$ by attaching a single pendant edge to one or more vertices of degree two. The proof of this result proceeds exactly as in the proof in Section 2 of Theorem 1, but requires a tedious case by case analysis of the exceptional graphs in the induction step. Similar results can likely be obtained for four or five vertices of degree less than three, if one can determine the exceptions and has the perseverance to carry out the case by case analysis. The following question, however, is open.

Conjecture. *Let k be any nonnegative integer. With finitely many exceptions, every simple graph having at most k vertices of degree less than three has two cycles whose lengths differ by one or two.*

Concerning Theorem 2, the 3-connectedness requirement is necessary. If it is assumed instead that G is 2-connected and, in addition, that every vertex has degree at least d, the corresponding statement is false. The counterexample (for $d = 3$) in Figure 1 has only cycles of lengths 4, 6, 9, 11, 13 and 15. This example can easily be generalized to an infinite family of counterexamples by attaching a sufficiently large odd number of copies of $K_{d,d} - e$ in a ring, as in the figure.

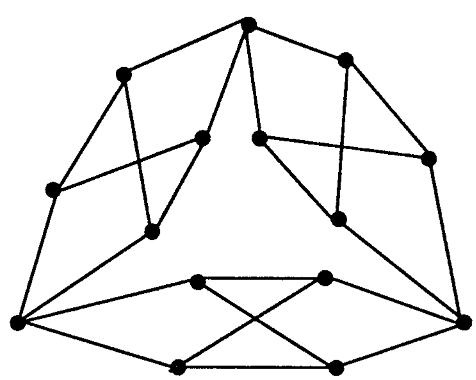

FIGURE 1. A counterexample.

A natural extension of Theorem 2 would be a condition guaranteeing the existence of cycles of three more more consecutive lengths. The only nonbipartite 3-connected graphs that we know of without cycles of three consecutive lengths are K_4 and the Petersen graph.

Problem. *Does there exist a function $f(k)$ such that every nonbipartite 3-connected graph with minimum degree at least $f(k)$ contains cycles of k consecutive lengths?*

2. THE PROOFS

The proofs of Theorems 1 and 2 rely on the two lemmas which are based on an approach of Thomassen and Toft [4] to nonseparating cycles in graphs. The terminology of ''bridges'' is originally due to Tutte [5] (also see Bondy and Murty [1], Voss [6] or West [7]). Let C be a cycle in a graph. A *C-bridge* is either (1) an edge not in C whose endpoints are in C or (2) a component of $G - V(C)$ together with the edges (and *vertices of attachment*) that connect it to C. The vertices of the bridge that are not vertices of attachment are called *internal vertices*. Two C-bridges A and B are said to *conflict* if either (1) they have three common vertices of attachment or (2) there are four vertices v_1, v_2, v_3, v_4 in cyclic order on C such that v_1, v_3 are vertices of attachment of A and v_2, v_4 are vertices of attachment of B. Two C-bridges that do not conflict are said to *avoid* each other.

Lemma 1. *Let G be a 2-connected graph, not a cycle, and let C be an induced cycle in G some bridge B of which has as many internal vertices are possible. Then either*

(1) *B is the only C-bridge, or else*
(2) *B is a C-bridge with exactly two vertices of attachment, and every other C-bridge is a path having the same two vertices of attachment to C as B.*

Proof. Let B' be any C-bridge different from B. Denote by S and S' the sets of vertices of attachment of B and B', respectively, to C. Because G is 2-connected, $|S| \geq 2$ and $|S'| \geq 2$.

Let $x, y \in S'$ and let P' be an xy-path in B'. We claim that $S = \{x, y\}$, and hence that $S' = \{x, y\}$ too. Suppose, to the contrary, that $S \neq \{x, y\}$, and consider $z \in S \setminus \{x, y\}$. Denote by P the xy-segment of C not including z. Since $P \cup P'$ is a cycle, there is an induced cycle C' with $V(C') \subseteq V(P \cup P')$. One of the bridges of C' has as internal vertices all the internal vertices of B and, in addition, the vertex z, contradicting the maximality hypothesis on C. We conclude that $S = \{x, y\}$.

Similar reasoning shows that neither $B' - x$ nor $B' - y$ contains an induced cycle. Therefore $B' - \{x, y\}$ is a tree having at most one vertex adjacent to each of x and y. It readily follows from the 2-connectedness of G that $B' = P'$. ∎

Lemma 2. *Let G be a nonbipartite 2-connected graph, not a cycle, and let C be an induced odd cycle in G some bridge B of which has as many internal vertices as possible. Then every other bridge of C is bipartite and avoids B.*

Proof. Let B' be a C-bridge different from B. Denote by S and S' the sets of vertices of attachment of B and B', respectively, to C. Because G is 2-connected, $|S| \geq 2$ and $|S'| \geq 2$.

Let $x, y \in S'$. Neither $B' - x$ nor $B' - y$ contains an induced odd cycle because one of the bridges of such a cycle would have as internal vertices all the internal vertices of B as well as x or y, respectively, contradicting the maximality hypothesis on C. The connectedness of $B' - \{x, y\}$ now implies that B' itself is bipartite.

Let P' be any xy-path in B', and let P be the xy-path in C for which the cycle $P \cup P'$ is odd. Let C' be an induced odd cycle with $V(C') \subseteq V(P \cup P')$. The cycle C' has a bridge whose internal vertices include all the internal vertices of B. By the maximality hypothesis on C, one deduces that $S \subseteq V(P)$.

If x and y are the only vertices of attachment of B' to C, then it is clear that B and B' avoid one another, as claimed. Therefore, suppose that $|S'| \geq 3$, and let the vertices of S', in cyclic order on C, be $v_1, v_2, \ldots, v_m$. For $1 \leq i \leq m$, denote by P_i the $v_i v_{i+1}$-segment of C, and let P_i' be a $v_i v_{i+1}$-path in B' (indices modulo m). Because C is odd, $\sum_{i=1}^{m} |E(P_i)|$ is odd, and because B' is bipartite, $\sum_{i=1}^{m} |E(P_i')|$ is even. It follows that the cycle $P_i \cup P_i'$ is odd for some $i, 1 \leq i \leq m$. As noted above, this implies that $S \subseteq V(P_i)$. Thus B and B' avoid one another. ∎

Proof of Theorem 1. The proof is by induction on the number n of vertices in the graph G. Theorem 1 is vacuously true for $n \leq 3$. Let G be a graph on $n > 3$ vertices and assume the theorem is true for graphs with fewer than n vertices.

By considering an appropriate block of G, we may suppose that G is 2-connected. Let C and B be as described in Lemma 1. By this lemma, if there is a C-bridge other than B, each such bridge is necessarily a path; moreover, all C-bridges, including B, have the same two vertices of attachment x, y. Every other vertex of C is thus of degree two. Because G is assumed to have at most two vertices of degree less than three, the length of C is three or four. In the former case, there is exactly one path bridge, of length two; in the latter case, x and y are nonconsecutive vertices of C and there is exactly one path bridge, of length one. In both cases, G has cycles of lengths three and four. Therefore, it may be assumed that B is the only C-bridge.

There can be at most two vertices on C that are not vertices of B. Hence, unless C is a 4-cycle with alternate vertices in B, there exist two vertices of attachment of B to C, say x and y, for which the two xy-paths in C differ in length by one or two. Denote these two xy-paths of C by P_1 and P_2, and let P be an xy-path in B. The cycles $P \cup P_1$ and $P \cup P_2$ differ in length by one or two, which proves the theorem.

In the case that C is a 4-cycle to which B has two vertices of attachment, the bridge B itself is a simple graph in which every vertex, except possibly its two vertices of attachment to C, has degree at least three. By induction, either the theorem is proved or B is K_1, clearly impossible, or K_2, impossible because C is induced. ∎

Proof of Theorem 2. Let C and B be as described in Lemma 2. We claim that B is the only bridge of C. By way of contradiction, suppose that C has a second bridge B'. By Lemma 2, the bridges B and B' avoid one another. There is thus a segment of C, connecting two consecutive vertices of attachment x, y of B to C, which contains every vertex of attachment of B' to C (and no other vertex of attachment of B to C). But then $\{x, y\}$ is a 2-vertex cut of G, contradicting the hypothesis of 3-connectedness. Thus B is indeed the only bridge of C.

Since C is odd, there exist two vertices of C, say x and y, for which the two xy-segments of C, P_1 and P_2, differ in length by one. Moreover, because B is the only bridge of C, x and y are vertices of attachment of B to C. Let P be an xy-path in B. The cycles $P \cup P_1$ and $P \cup P_2$ differ in length by one. ∎

Remarks. The fact used in the proof of Theorem 2, that G contains an odd cycle C with only one bridge B, is also an immediate corollary of a theorem of Tutte stating that the induced nonseparating cycles in a 3-connected graph generate its cycle space [4].

Since this paper was accepted for publication, A. K. Kelmans has drawn the authors' attention to an article of his [3] in which nonseparating cycles are treated and in which results of a similar nature to our Lemma 1 may be found.

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