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Computing the Discrete Fourier Transform on a Hexagonal Lattice

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Abstract The computation of the Discrete Fourier Transform for a general lattice in $\mathbb{R}^d$ can be reduced to the computation of the standard 1-dimensional Discrete Fourier Transform. We provide a mathematically rigorous but simple treatment of this procedure and apply it to the DFT on the hexagonal lattice.

Keywords Discrete Fourier transform · Hexagonal lattice

1 Introduction

Traditional image processing algorithms are usually carried out on rectangular arrays, but there is a growing research literature on image processing using other sampling grids [1, 4, 10, 12, 13]. Of particular interest is sampling on a hexagonal grid. Hexagonal grids provide for higher packing density, give a more accurate approximation of circular regions, and exhibit symmetric neighbor adjacency (i.e. the distance from the center of any hexagon to the center of any adjacent hexagon is the same). An extensive list of references can be found in [11]. One of the fundamental tools in image processing is the discrete Fourier transform (DFT) [14]. It is the intention of this paper to present a straightforward approach to the DFT for a general lattice L in $\mathbb{R}^d$, an approach perhaps more direct than previous treatments. The spatial domain of the DFT in this context is a set of coset representatives of the quotient of L by a sublattice of L. The computation of the DFT, even in this general case, can efficiently and easily be reduced to the computation of the standard 1-dimensional DFT. This result is then applied to hexagonal arrays, in particular to multiresolution arrays that allow for fast "zooming in" to view fine image detail or "zooming out" to view global image features.

Previous approaches to the DFT on a hexagonal grid include [3, 7], in which the DFT is converted to a square grid, and [15] in which the GBT (generalized balanced ternary) system for indexing a hexagonal grid is used. Our approach to the DFT uses the Smith normal form of a square integer matrix. As pointed out by one of the referees, the use of this normalized form appeared previously in Problem 20, Chap. 2, of [1] and in [6]. This paper contributes a mathematically rigorous yet simple approach before applying it to the hexagonal lattice.

This introductory section contains general background on the DFT. Section 2 concerns the spatial and frequency domains of the DFT for a general lattice. The result on reducing the computation of the lattice DFT to the standard DFT appears in Sect. 3. Section 4 applies the results to the hexagonal grid.

In dimension 1, grid cells are unit intervals centered say, at the points $0, \pm 1, \pm 2, \dots$. An image can be thought of as a complex valued function defined on a finite subset, say $\{0, 1, 2, \dots, N-1\}$, of these points. Let $\mathbb{C}^{[N]}$ denote the vector space of all such functions. The discrete Fourier transform is the linear transformation $\mathcal{F} : \mathbb{C}^{[N]} \to \mathbb{C}^{[N]}$ defined by

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$$(\mathcal{F}a)(k) = \sum_{j=0}^{N-1} a(j) e^{-2\pi i \frac{jk}{N}}.$$

The general mathematical setting for the DFT is a finite Abelian group G of order, say N. The group operation will be denoted $+$. Let $\widehat{G}$ denote the *character group* $\mathrm{Hom}(G, T_N)$ of G, i.e., the group of all homomorphisms of G into the multiplicative group T_N of all Nth roots of unity. The addition on $\widehat{G}$ is defined by $(\chi + \psi)(g) = \chi(g)\psi(g)$ for all $\chi, \psi \in \mathrm{Hom}(G, T_N)$. In signal processing terminology, G is the *spatial domain* and $\widehat{G}$ the *frequency domain*. Any signal can be fully described in either of these domains. The Fourier transform is the tool that allows us to go back and forth between the two. Depending on what we want to do with the signal, one domain tends to be more useful, or mathematically simpler, than the other. It is a standard result that $\widehat{G} \cong G$, so the spatial and frequency domains are algebraically the same. In particular $N = |G| = |\widehat{G}|$. In the context of this paper, as developed in Sect. 2, the situation is more concrete. Both the spatial domain G and the frequency domain $\widehat{G}$ will consist of a finite set of cells of some grid in Euclidean space $\mathbb{R}^n$.

Let $\mathbb{C}^G$ and $\mathbb{C}^{\widehat{G}}$ denote the vector spaces of complex-valued functions on G and $\widehat{G}$, respectively. From an image processing point of view, a function in $\mathbb{C}^G$ is merely an image. Both $\mathbb{C}^G$ and $\mathbb{C}^{\widehat{G}}$ are inner product spaces, the inner product on $\mathbb{C}^G$ being

$$\langle a, b\rangle = \sum_{g\in G} a(g)\overline{b(g)},$$

and associated norm

$$\|a\| = \sqrt{\langle a, a\rangle} = \left(\sum_{g\in G} |a(g)|^2\right)^{1/2}.$$

The inner product and norm on $\mathbb{C}^{\widehat{G}}$ are similarly defined. The Discrete Fourier Transform (DFT) is the linear transformation

$$\mathcal{F}\colon \quad \begin{aligned} &\mathbb{C}^G \to \mathbb{C}^{\widehat{G}} \\ &a \mapsto \widehat{a} \end{aligned}$$

defined by

$$\widehat{a}(\chi) = \frac{1}{\sqrt{N}} \sum_{g\in G} a(g)\overline{\chi(g)} \tag{1}$$

for all $\chi \in \widehat{G}$. Moreover, $\mathcal{F}$ is an isometry, that is, $\|\mathcal{F}(a)\| = \|a\|$ for all $a \in \mathbb{C}^G$. The inverse Fourier transform

$$\mathcal{F}^{-1}\colon \quad \begin{aligned} &\mathbb{C}^{\widehat{G}} \to \mathbb{C}^G \\ &\widehat{a} \mapsto a \end{aligned}$$

is defined by

$$a(g) = \frac{1}{\sqrt{N}} \sum_{\chi\in\widehat{G}} \widehat{a}(\chi)\chi(g) \tag{2}$$

for all $g \in G$, the basic result being that $\mathcal{F}^{-1} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{F}^{-1} = \mathrm{id}$.

2 Spatial and Frequency Domains for a Lattice

Practical considerations dictate that the cells of a sampling grid should be translates of a single cell to the points of a lattice L in $\mathbb{R}^d$. In dimension 2, the most commonly utilized lattices are the square lattice and the hexagonal lattice. A *d-dimensional lattice* is an integer linear combination of d linearly independent vectors in $\mathbb{R}^d$. These d vectors are called a set of *generators* of the lattice. The matrix whose rows are the generators of L is called the *generator matrix* of L. Let $L_0 < L$ denote that L_0 is a d-dimensional sublattice of d-dimensional lattice L. In this case the quotient L/L_0 of the Abelian groups L and L_0 is a finite Abelian group.

An *image* is a function defined on a finite subset D of L. In order to be amenable to the DFT, the set D should have a natural Abelian group structure, natural in the sense that the group operation be compatible with the vector addition on L. This is the case when there exists a sublattice L_0 of L such that D is a set of coset representative of the quotient L/L_0. Given L and D there is an efficient algorithm [8] for deciding whether or not there is such a sublattice L_0 for which D is a set of coset representatives of L/L_0 and, if so, for finding generators for the lattice L_0. In all that follows we will assume that our domain D is such a set. The group L/L_0 will serve as the spatial domain of the DFT.

Note that, in the 1-dimensional case, $L = \mathbb{Z}$ and $L_0 = \{Ni : i \in \mathbb{Z}\}$ for some integer N. The group L/L_0 is the group of integers modulo N, and its character group $\widehat{L/L_0}$ consists of the homomorphisms

$$j \mapsto e^{2\pi i \frac{jk}{N}}$$

for $k = 0, 1, \ldots, N-1$. So the DFT in this case is just the standard DFT.

Given a d-dimensional lattice L and sublattice L_0, call a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ of generators of L an *elementary set of generators* for L *with respect to sublattice* L_0 if, for some integers $N_1, N_2, \ldots, N_d$,

$$S = \{j_1\mathbf{v}_1 + j_2\mathbf{v}_2 + \cdots + j_d\mathbf{v}_d : 0 \le j_i < N_i, 0 \le i \le d\} \tag{3}$$

is a set of coset representatives of L/L_0. In this case call the integers $N_1, N_2, \ldots, N_d$ *divisors* of L/L_0.

Lemma 1 *If $L_0 < L$, then L has an elementary set of generators with respect to L_0.*

The proof of Lemma 1 is subsumed in the Algorithm below, whose output is an elementary set of generators for L with respect to L_0 and the corresponding set of divisors of L/L_0. It is well known [2] that any square matrix A with integer entries can be put in diagonal form (Smith Normal Form) using a sequence of the following row and column operations.

Interchange two rows (columns)
Multiply a row (column) by ± 1
Add an integer multiple of one row (column) to another row (column)

There is a standard algorithm, which we omit, for accomplishing this. Any such row operation is given by left multiplication of A by an *elementary matrix* U on the left and any column operation by multiplication of A by an elementary matrix on the right.

Algorithm Input: lattices $L_0 < L$ and any generator matrix M of L.

Output: a matrix M_L whose rows comprise an elementary set of generators of L with respect to L_0, and a diagonal matrix E whose diagonal entries are a corresponding set of divisors of L/L_0.

Since generators of L_0 are integer linear combinations of the generators of L, there is an easily computed integer matrix A such that AM is a generator matrix of L_0.

Let $V_1, V_2, \ldots, V_s$ and $U_1, U_2, \ldots, U_t$ be elementary matrices (from the Smith Normal form) used to diagonalize A. Thus the following matrix E is diagonal:

$$E = V_s \cdots V_2 V_1 A U_1 U_2 \cdots U_t.$$

Let

$$M_L = (U_1 U_2 \cdots U_t)^{-1} M.$$

Proof (of the validity of the Algorithm) Let $U = U_1 U_2 \cdots U_t$ and $V = V_s \cdots V_2 V_1$. Since the rows of AM are the generators of L_0 and since V is unimodular, the rows of $M_{L_0} = V(AM)$ are again a set of generators of L_0. (By a *unimodular* matrix, we mean a square matrix with integer entries and determinant $+1$ or -1.) Likewise the rows of $M_L = U^{-1}M$ are a set of generators of L. Therefore

$$EM_L = (VAU)(U^{-1}M) = V(AM) = M_{L_0}, \tag{4}$$

which says that the rows of M_L comprise an elementary set of generators of L with respect to L_0 with corresponding divisors on the diagonal of E. □

It follows immediately from Lemma 1 and the Algorithm, in particular the formula for matrix E, that, if M is a generator matrix of L and AM a generator matrix of L_0, then

$$|L/L_0| = N_1 N_2 \cdots N_d = |\det A|. \tag{5}$$

Note also that the divisors $N_1, N_2, \ldots, N_d$ are not necessarily unique, but according to the theory of finite Abelian groups, can be made unique up to order under the additional divisibility condition $N_1 | N_2 | \cdots | N_d$. In this case the divisors are called *elementary divisors*.

Example As a simple example, let L be the square lattice in $\mathbb{R}^2$ generated by $(1, 0)$ and $(0, 1)$, and let L_0 be the sublattice generated by $(2, 4)$ and $(6, 16)$. Then

$$A = \begin{pmatrix} 2 & 4 \\ 6 & 16 \end{pmatrix},$$

$$E = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} A \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix},$$

$$M_L = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

Then an elementary set of generators for L with respect to L_0 is $\{(1, 2), (0, 1)\}$ with corresponding set $\{2, 4\}$ of divisors. Therefore

$$\{(0, 0), (0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (1, 4), (1, 5)\}$$

is a set of coset representatives of L/L_0.

For a lattice L, the geometric *dual* (or *reciprocal lattice*) of L is defined by

$$L^* = \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{y} \rangle \text{ is an integer for all } \mathbf{x} \in L\},$$

where $\langle \mathbf{x}, \mathbf{y} \rangle$ is the standard Euclidean inner product. If L_0 is a sublattice of L, then the first two of the following properties are well known, and the third is easy to check. Here $\cong$ denotes group isomorphism and M^{-T} the transpose of the inverse of M.

1. $L^* < L_0^*$
2. $\widehat{L/L_0} \cong L_0^*/L^*$
3. If M is a generator matrix of L, then M^{-T} is a generator matrix of L^*

Since the quotient L/L_0 serves as the spatial domain of the DFT, the quotient L_0^*/L^*, in view of property (2), serves as the frequency domain. Consider the 1-dimensional example where, for some integer N, we take $L = \mathbb{Z}$, $L_0 = N\mathbb{Z}$, and $\{0, 1, \ldots, N-1\}$ is a set of coset representatives of L/L_0. Then $L^* = Z$, $L_0^* = (\frac{1}{N})\mathbb{Z}$, and $\{0, \frac{1}{N}, \frac{2}{N}, \ldots, \frac{N-1}{N}\}$ is a set of coset representatives of L_0^*/L^*. Another example appears below, and the situation for the 2-dimensional hexagonal lattice appears in Sect. 4.

Concerning the situation in the frequency domain, (4) and property (3) above imply that $M_{L^*} := M_L^{-T}$ is a generator matrix of L^*, $M_{L_0^*} := M_{L_0}^{-T}$ is a generator matrix of L_0^*, and, taking the inverse transpose of each side of equation (4),

$$EM_{L_0^*} = M_{L^*}.$$

This implies that the rows of $M_{L_0^*}$ comprise an elementary set of generators for L_0^* with respect to L^*. Moreover, the corresponding divisors are the same as for L with respect to L_0.

Example This is a continuation of the example that appears earlier in this section in which L is the square lattice. In this case L^* is again the square lattice and, as stated above,

$$M_{L_0^*} := M_{L_0}^{-T} = (EM_L)^{-T} = \left[\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\right]^{-T}$$
$$= \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{4} \end{pmatrix}$$

Then an elementary set of generators for L_0^* with respect to L^* is $\{(\frac{1}{2}, 0), (-\frac{1}{2}, \frac{1}{4})\}$ with corresponding set $\{2, 4\}$ of divisors. Therefore

$$\Big\{(0,0), \left(-\frac{1}{2}, \frac{1}{4}\right), \left(-1, \frac{1}{2}\right), \left(-\frac{3}{2}, \frac{3}{4}\right),$$
$$\left(\frac{1}{2}, 0\right), \left(0, \frac{1}{4}\right), \left(-\frac{1}{2}, \frac{1}{2}\right), \left(-1, \frac{3}{4}\right)\Big\}$$

is a set of coset representatives of L/L_0.

3 The DFT on a Lattice

Given $L_0 < L$ and $g \in L$, let $\overline{g}$ denote the coset of g in L/L_0, and likewise given $h \in L_0^*$, let $\overline{h}$ denote the coset of h in L_0^*/L^*. The isomorphism

$$\phi: \quad L_0^*/L^* \to \widehat{L/L_0}$$
$$\overline{h} \mapsto \chi_{\overline{h}}$$

in property (2) in Sect. 2 is given by

$$\chi_{\overline{h}}(\overline{g}) = e^{2\pi i \langle g,h\rangle}$$

for $\overline{g} \in L/L_0$, the right hand side in the equation being independent of the choice of the particular coset representatives g and h. In light of this and the DFT formula (1), the DFT with spatial domain $G = L/L_0$, frequency domain $G^* = L_0^*/L^*$, and order $N = |G| = |G^*|$, is the linear transformation

$$\mathcal{F} : \mathbb{C}^G \to \mathbb{C}^{G^*}$$

given by

$$(\mathcal{F}a)(\overline{h}) = \frac{1}{\sqrt{N}} \sum_{\overline{g} \in G} a(\overline{g}) e^{-2\pi i \langle g,h\rangle} \tag{6}$$

for all $a \in \mathbb{C}^G$ and all $\overline{h} \in G^*$. The inverse Fourier transform is given by

$$(\mathcal{F}^{-1}a^*)(\overline{g}) = \frac{1}{\sqrt{N}} \sum_{\overline{h} \in G^*} a^*(\overline{h}) e^{2\pi i \langle g,h\rangle} \tag{7}$$

for all $a^* \in \mathbb{C}^{G^*}$ and all $\overline{g} \in G$. The following theorem and its proof give a method for reducing the calculation of the DFT (6) and inverse DFT (7) to the calculation of 1-dimensional DFTs.

Theorem 2 *If the quotient L/L_0 has divisors $N_1, N_2, \ldots, N_d$, then the general DFT or inverse DFT of formulas (6) and (7) can be reduced to computing 1-dimensional DFTs of the form*

$$\widehat{a}(k) = \sum_{j=0}^{M-1} a(j) e^{-2\pi i \frac{jk}{M}}.$$

The precise form of the DFT is given in (9) and (10) below. Using a fast Fourier transform on each 1-dimensional DFT, the above result provides a run time of

$$O\left(N \sum_{i=1}^{d} \frac{\log N_i}{N_1 N_2 \cdots N_{i-1}}\right).$$

Proof We prove the result for the DFT; a similar proof holds for the inverse DFT. Since the sum on the right hand side of (6) is independent of the particular set of representatives, we take a set S of representatives of the form (3) associated with an elementary set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_d\}$ of generators for L with respect to L_0. If these vectors form the rows of generator matrix M, then by property (3) in Sect. 2, the rows $\{\mathbf{v}_1^*, \mathbf{v}_2^*, \ldots, \mathbf{v}_d^*\}$ of $(EM)^{-T}$ comprise an elementary set of generators for L_0^* with respect to L^*, where E is the diagonal matrix in the Algorithm of Sect. 2, with diagonal elements $N_1, N_2, \ldots, N_d$. Note that, for the matrix of inner products, we have

$$(\langle \mathbf{v}_j, \mathbf{v}_k^* \rangle) = (EM)^{-T} M^T = E^{-1}.$$

Hence

$$\langle \mathbf{v}_j, \mathbf{v}_k^* \rangle = \begin{cases} \frac{1}{N_j}, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

So

$$\left\langle \sum_{i=1}^{d} j_i \mathbf{v}_i, \sum_{i=1}^{d} k_i \mathbf{v}_i^* \right\rangle = \sum_{i=1}^{d} \frac{1}{N_i} j_i k_i. \tag{8}$$

For ease of notation use $J=(j_1,j_2,\dots,j_d)$ for $j_1\mathbf{v}_1+j_2\mathbf{v}_2+\cdots+j_d\mathbf{v}_d$ and $K=(k_1,k_2,\dots,k_d)$ for $k_1\mathbf{v}_1^*+k_2\mathbf{v}_2^*+\cdots+k_d\mathbf{v}_d^*$. Then using formula (8) the DFT in formula (6) becomes

$$\begin{aligned}(\mathcal{F}a)(K) &= \frac{1}{\sqrt{N}}\sum a(J)e^{-2\pi i\frac{j_1k_1}{N_1}}e^{-2\pi i\frac{j_2k_2}{N_2}}\cdots e^{-2\pi i\frac{j_dk_d}{N_d}}\\ &= \frac{1}{\sqrt{N}}\sum_{j_d=0}^{N_d-1}e^{-2\pi i\frac{j_dk_d}{N_d}}\\ &\quad\times\left(\cdots\left(\sum_{j_2=0}^{N_2-1}e^{-2\pi i\frac{j_2k_2}{N_2}}\left(\sum_{j_1=0}^{N_1-1}a(J)e^{-2\pi i\frac{j_1k_1}{N_1}}\right)\right)\right), \end{aligned}\tag{9}$$

where the summation in the first line is $0\le j_1<N_1,\dots,0\le j_d<N_d$. Thus the DFT can be reduced to computing the following 1-dimensional DFTs, the first done N/N_1 times, the second $N/(N_1N_2)$ times, …, the last computed just once.

$$\begin{aligned}b_1(k_1,j_2,\dots,j_d) &= \sum_{j_1=0}^{N_1-1}a(j_1,j_2,\dots,j_d)e^{-2\pi i\frac{j_1k_1}{N_1}},\\ b_2(k_1,k_2,j_3,\dots,j_d) &= \sum_{j_2=0}^{N_2-1}b_1(k_1,j_2,\dots,j_d)e^{-2\pi i\frac{j_2k_2}{N_2}},\\ &\dots\\ (\mathcal{F}a)(K) &= b_d(k_1,k_2,\dots,k_d)\\ &= \sum_{j_d=0}^{N_d-1}b_{d-1}(k_1,k_2,\dots,k_{d-1},j_d)e^{-2\pi i\frac{j_dk_d}{N_d}}.\end{aligned}\tag{10}$$

Since a fast Fourier transform running time on a 1-dimensional DFT of size N_i is $O(N_i\log N_i)$, the total running time is as given in the statement of the theorem. □

Example This is a continuation of the example that appears twice in the previous section, where L is the square lattice in $\mathbb{R}^2$ generated by $(1,0)$ and $(0,1)$ and L_0 is the sublattice generated by $(2,4)$ and $(6,16)$. Let a be a function defined on L/L_0. As an example we compute $\widehat{a}(\overline{(\frac{1}{2},\frac{1}{2})})$. Note that $(\frac{1}{2},\frac{1}{2})$ and $(-\frac{1}{2},\frac{1}{2})=(\frac{1}{2},0)+2(-\frac{1}{2},\frac{1}{4})$ represent the same coset in L_0^*/L^* and, in the notation of Theorem 2, this is coset $K=(1,2)$. Then by (9) and (10) above

$$\widehat{a}(K)=\sum_{j_2=0}^{3}b(k_1,j_2)e^{-\pi i j_2},$$

where

$$\begin{aligned}b(k_1,0)&=a(0,0)-a(1,0),\\ b(k_1,1)&=a(0,1)-a(1,1),\\ b(k_1,2)&=a(0,2)-a(1,2),\\ b(k_1,3)&=a(0,3)-a(1,3).\end{aligned}$$

In terms of actual coordinates in $\mathbb{R}^2$,

$$\begin{aligned}\widehat{a}\overline{\left(\frac{1}{2},\frac{1}{2}\right)} &= a(0,0)+a(1,3)+a(0,2)+a(1,5)\\ &\quad -a(1,2)-a(0,1)-a(1,4)-a(0,3).\end{aligned}$$

4 Hexagonal Grids

The *standard hexagonal lattice* H is the lattice in $\mathbb{R}^2$ with generators

$$\mathbf{v}_1=(1,0),\qquad \mathbf{v}_2=\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right).$$

Any scaled and/or rotated copy of H is also called *hexagonal*. The dual lattice H^* is again hexagonal, with generators

$$\mathbf{v}_1^*=\left(1,\frac{1}{\sqrt{3}},\right),\qquad \mathbf{v}_2^*=\left(0,\frac{2}{\sqrt{3}}\right).$$

4.1 Hexagonal Domains

A particularly nice sampling method is to take a hexagonal shaped spatial domain D. Figure 1 shows representatives of two (of the many) such families. In general, it is not hard to show that there exists such a hexagonal shaped set D of coset representatives of H/H_0 in the case that H_0 is also a hexagonal lattice. Call H/H_0 a *hexagonal domain* if both H and H_0 are hexagonal lattices. If M is the generator matrix of H with $\mathbf{v}_1,\mathbf{v}_2$ as the rows, then it is easy to show that a matrix is the generator matrix of a hexagonal sublattice of H if and only if it has the form AM, where

$$A=\begin{pmatrix} r & s\\ -s & r-s\end{pmatrix}$$

for some integers r,s. This is because, if $(r,s)M$ is an arbitrary point of H, then $(-s,r-s)M$ is just $(r,s)M$ rotated $2\pi/3$. Let $H_{r,s}$ be the sublattice of H with generator matrix AM. According to formula (5) the order N of $H/H_{r,s}$ is

$$N=|H/H_{r,s}|=|\det A|=|r^2-rs+s^2|.$$

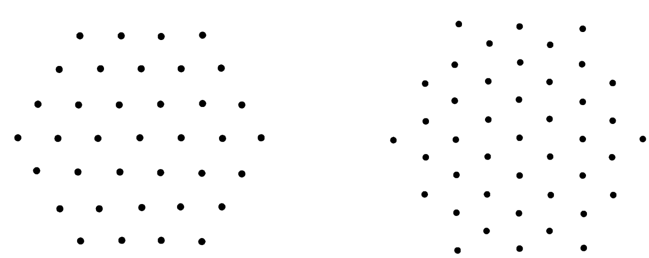

Fig. 1 Hexagonal domains

Since H and $H_{r,s}$ are both hexagonal, so are H^* and $H^*_{r,s}$. Hence both the spatial domain $G_{r,s} = H/H_{r,s}$ and frequency domain $G^*_{r,s} = H^*_{r,s}/H^*$ are hexagonal in this case. The next result shows that the DFT often reduces to a single 1-dimensional DFT.

Theorem 3 *If r and s are relatively prime, then there are isomorphisms $G_{r,s} \cong \mathbb{Z}/N\mathbb{Z}$ and $G^*_{r,s} \cong \mathbb{Z}/N\mathbb{Z}$ for both the spatial and frequency domains, and with respect to these isomorphisms, the discrete Fourier transform $\mathcal{F}: \mathbb{C}^{\mathbb{Z}/N\mathbb{Z}} \to \mathbb{C}^{\mathbb{Z}/N\mathbb{Z}}$ is given by*

$$(\mathcal{F}a)(k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} a(j) e^{-2\pi i \frac{jk}{N}}.$$

Moreover, $\overline{\mathbf{v}_1} = (1, 0)$ is a generator of the cycle group $G_{r,s}$ that corresponds to the generator $\overline{1} \in \mathbb{Z}/N\mathbb{Z}$ under the isomorphism $G_{r,s} \cong \mathbb{Z}/N\mathbb{Z}$.

Proof It is first shown, for some $\mathbf{w} \in H$, that $\{\mathbf{v}_1, \mathbf{w}\}$ is an elementary generating set for H with respect to $H_{r,s}$, with corresponding elementary divisors N and 1. The order of the group $G_{r,s}$ is $N = |r^2 - rs + s^2|$. Therefore $N\mathbf{v}_1 \in H_{r,s}$. Since r and s are relatively prime, there exist integers x, y such that $sx + (r - s)y = 1$. Let $c = rx - sy$. Then $c\mathbf{v}_1 + \mathbf{v}_2 = (c, 1)M = (x, y)AM \in H_{r,s}$. With $\mathbf{w} = c\mathbf{v}_1 + \mathbf{v}_2$, clearly $\{\mathbf{v}_1, \mathbf{w}\}$ generates H. In fact, this is an elementary generating set with corresponding elementary divisors N and 1 because, first, $|S| = |\{j\mathbf{v}_1 + \mathbf{w} : 0 \le j < N\}| = N = |G_{r,s}|$ and, second, both $N\mathbf{v}_1 \in H_{r,s}$ and $\mathbf{w} \in H_{r,s}$. We have thus shown that $G_{r,s}$ is a cycle group of order N with generator $\mathbf{v}_1$.

Now $\{0, \mathbf{v}_1, 2\mathbf{v}_1, \ldots, (N-1)\mathbf{v}_1\}$ is a set of coset representatives of $G_{r,s}$. According to the discussion at the end of Sect. 2, there is an elementary set $\{\mathbf{v}^*, \mathbf{w}^*\}$ of generators of $G^*_{r,s}$ with respect to G^*, so that $N\mathbf{v}^*, \mathbf{w}^* \in H^*$, and hence $\{0, \mathbf{v}^*, 2\mathbf{v}^*, \ldots, (N-1)\mathbf{v}^*\}$ is a set of coset representatives of $G^*_{r,s}$. Hence $G^*_{r,s}$ is also a cycle group of order N. Moreover, by formula (9) we have

$$(\mathcal{F}a)(k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} a(j) e^{-2\pi i \frac{jk}{N}}.$$

□

4.2 A Particular Hexagonal Domain

It is instructive to consider how Theorem 3 applies in the case of the family of domains, one member of which is shown in the left diagram of Fig. 1. In general this family of domains is given by

$$D_n = \{j\mathbf{v}_1 + k\mathbf{v}_2 : |j| \le n, |k| \le n, |j - k| \le n\}.$$

It is not hard to count the number of lattice points in D_n:

$$N = |D_n| = 3n^2 + 3n + 1.$$

The next result states that D_n is, indeed, a set of coset representatives of the hexagonal quotient $H/H_{r,s}$ where $r = 2n + 1$ and $s = n + 1$. Since $2n + 1$ and $n + 1$ are relatively prime, the DFT on D_n, according to Theorem 3, can be reduced to the computation of a single, ordinary 1-dimensional DFT. What is required to make this practical, however, is an effective correspondence between the set D_n points of the spatial domain and the integers $\{0, 1, 2, \ldots, 3n^2 + 3n\}$ in $\mathbb{Z}/N\mathbb{Z}$. This is also provided in the next result.

Theorem 4 *The set D_n is a set of coset representatives of $H/H_{2n+1,n+1}$. Moreover, the isomorphism $\phi: D_n \to \mathbb{Z}/N\mathbb{Z}$ of Theorem 3 is given by*

$$\phi(j\mathbf{v}_1 + k\mathbf{v}_2) \equiv j + k(n+1) \pmod{N}.$$

Proof Concerning the first statement, since $|H/H_{2n+1,n+1}| = |\det A| = 3n^2 + 3n + 1 = |D_n|$, to show that D_n is a set of coset representatives, it suffices to show that no two elements of D_n represent the same coset. Let $H' = H_{2n+1,n+1}$ and suppose that $d, d' \in D_n$ represent the same coset, i.e. $d - d' \in H'$. It is easy to verify that $d - d' \in D_{2n}$. Hence $d - d' \in H' \cap D_{2n}$. The six closest points to the origin in H' are all at a distance $\sqrt{N}$; they are

$$\pm((2n+1)\mathbf{v}_1 + n\mathbf{v}_2),$$

$$\pm((n+1)\mathbf{v}_1 + (2n+1)\mathbf{v}_2),$$

$$\pm((n+1)\mathbf{v}_1 - n\mathbf{v}_2).$$

According to the definition, none of these points lie in D_{2n}. The next smallest distance from the origin to a point in H' is $\sqrt{3N}$. But the furthest distance from the origin to a point in D_{2n} is $2n < \sqrt{3N}$. Therefore $H' \cap D_{2n} = \{0\}$, which implies that $d - d' = 0$ and $d = d'$.

Concerning the bijection ϕ, let $\phi(j\mathbf{v}_1 + k\mathbf{v}_2) = c \in \mathbb{Z}/N\mathbb{Z}$. Since $\phi(\mathbf{v}_1) = 1$ by Theorem 3, in the notation of Sect. 2, $(j, k)M - (c, 0)M \in H'$. This is true if and only if $(j - c, k)M = \mathbf{x}AM$ for some $\mathbf{x} \in \mathbb{Z}^2$ where, by definition,

$$A = \begin{pmatrix} 2n+1 & n+1 \\ -(n+1) & n \end{pmatrix}.$$

This is equivalent to $(j - c, k) = \mathbf{x}A = \mathbf{x}V^{-1}EU^{-1}$ for some $\mathbf{x} \in \mathbb{Z}^2$, or equivalently

$$(j - c, k)U = (p, q)\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} = (pN, q)$$

for some $(p, q) \in \mathbb{Z}^2$. Since the unimodular matrix U is computed to be

$$U = \begin{pmatrix} n & -1 \\ n+1 & -1 \end{pmatrix},$$

$(j-c,k)U = (j-c)n + k(n+1), *)$. Hence there exists such a pair (p,q) if and only if $N \mid (j-c)n + k(n+1)$, which is the case when $c = j + k(n+1) \pmod N$. □

By the same methods, a result similar to that of Theorem 4 can be proved for the frequency domain. The vectors

$$u_1^* = \frac{n}{N} v_1^* + \frac{n+1}{N} v_2^*,$$

$$u_2^* = -\frac{n+1}{N} v_1^* + \frac{2n+1}{N} v_2^*$$

generate the dual lattice $H^*_{2n+1,n+1}$ and

$$D_n^* = \{j\mathbf{u}_1^* + k\mathbf{u}_2^* : |j| \le n, |k| \le n, |j+k| \le n\}$$

is a set of coset representatives of $(H/H_{2n+1,n+1})^*$. Moreover, the isomorphism $\phi : D_n^* \to \mathbb{Z}/N\mathbb{Z}$ of Theorem 3 is given by

$$\phi(j\mathbf{u}_1^* + k\mathbf{u}_2^*) \equiv j + k(3n+2) \pmod N.$$

Example Consider the simple case $n = 1$ where $N = 7$. The isomorphisms of D_1 and D_1^* with $\mathbb{Z}/7\mathbb{Z}$ are shown in Fig. 2. Note that the scale in the figure is not accurate; the distance between neighboring points in the left diagram is 1 whereas in the right diagram it is $2/\sqrt{21}$. Suppose that the DFT of a function a defined on $H/H_{3,2}$ is to be computed at the element of $(H/H_{3,2})^*$ represented by, say, the point $h^* = (-3/7, -1/7\sqrt{2})$, which corresponds to 4 in D_1^* of Fig. 2. According to Theorems 3 and 4

$$(\mathcal{F}a)(\overline{h^*}) = (\mathcal{F}a)(4) = \frac{1}{\sqrt{7}} \sum_{j=0}^{6} a(j) e^{-8\pi i j/7}.$$

We note that the 1-dimensional DFT for the family of hexagonal domains discussed in this section is not necessarily amenable to a Cooley-Tukey type fast Fourier transform because $N = 3n^2 + 3n + 1$ does not necessarily factor into small primes. In fact, 263 of the first 1000 integers of this form are themselves prime.

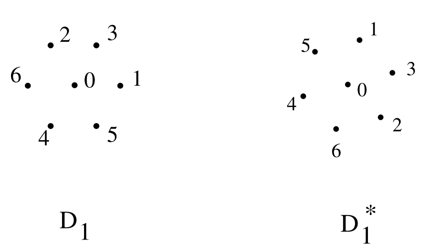

Fig. 2 Coset representatives of $H/H_{3,2}$ and of $(H/H_{3,2})^*$

4.3 Generalized Balanced Ternary

This example comes from applications of the DFT in the processing of multiresolution images on hexagonal grids [11]. For multiresolution image processing, a nested sequence of sampling domains is required, each domain the non-overlapping union of domains one level down in the sequence. If H_0 is a sublattice of the hexagonal lattice H, then such a nested sequence of domains is a sequence of coset representatives of quotients $G_n = H/H_n$, $n = 0, 1, 2, \dots$, where $H_n = H_0 A^n = \{\mathbf{x}A^n : \mathbf{x} \in H_0\}$ for some matrix A with $|\det A| > 1$. If S_n denotes a set of coset representatives of G_n, to see the nesting of $|\det A|$ disjoint translated copies of S_{n-1} in S_n, note that

$$S_n = \{s + tA^n : s \in S_{n-1}, t \in S_0\} = \bigcup_{t \in S_0} (tA^n + S_{n-1}).$$

The cases of practical interest occur when H/H_n is hexagonal and $|G_n|/|G_{n-1}| = |\det A|$ is small (the ratio in zooming in from one level to the next is small). Perhaps the most studied example, because of the elegant method of indexing its cells (lattice points) [9], is the Generalized Balanced Ternary (GBT) system [5, 9, 11]. The first two levels of the 2-dimensional GBT are shown in Fig. 3. Note that the second level is the non-overlapping union of 7 translated copies of the first level. We now consider the DFT for the 2-dimensional GBT, a topic dealt with in a more complicated way in [15].

Since it is notationally simpler, we will use complex numbers to denote lattice points. Let $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$. Then the hexagonal lattice H is generated by 1 and ω. Note that

$$1 + \omega + \omega^2 = 0.$$

The 2-dimensional GBT (generalized balanced ternary) hexagonal sublattices are defined by

$$H_n = \beta^n H = \{\beta^n \mathbf{x} : \mathbf{x} \in H\}, \quad \text{where } \beta = 2 - \omega.$$

The standard set D of coset representatives of H/H_1 consists of the origin and its six neighboring points in the hexagonal lattice H:

$$D = \{\epsilon_0 + \epsilon_1 \omega + \epsilon_2 \omega^2 : \epsilon_i = 0 \text{ or } 1\}.$$

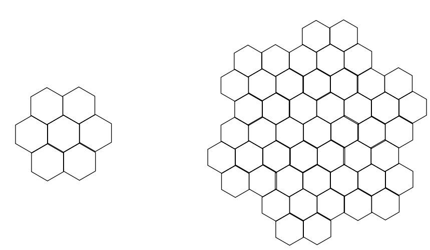

Fig. 3 GBT aggregates

It is more common in the literature to see the elements of D denoted $\{0,1,2,3,4,5,6\}$. The correspondence is $\epsilon_0+\epsilon_1\omega+\epsilon_2\omega^2\mapsto\epsilon_0+\epsilon_1 2+\epsilon_2 2^2$ (the rationale being that $\omega\equiv 2 \pmod{\beta}$). For example, 3 and $1+\omega$ denote the same lattice point. The standard set of coset representatives of H/H_n, $n\geq 1$, is

$$D_n=\left\{\sum_{k=0}^{n-1} d_k\beta^k : d_k\in D\right\}.$$

The order N of the group $\mathrm{GBT}_n=H/H_n$ is

$$N=|\mathrm{GBT}_n|=7^n.$$

The following theorem reduces the computation of the DFT for the GBT at any resolution n to a single 1-dimensional DFT. Since N is a power of 7, the DFT is highly amenable to a fast Fourier transform via the Cooley-Tukey algorithm.

Corollary 5 *There are isomorphisms* $\mathrm{GBT}_n\cong\mathbb{Z}/N\mathbb{Z}$ *and* $\mathrm{GBT}_n^*\cong\mathbb{Z}/N\mathbb{Z}$ *for both the spatial and frequency domains, and with respect to these isomorphisms, the discrete Fourier transform* $\mathcal{F}:\mathbb{C}^{\mathbb{Z}/N\mathbb{Z}}\to\mathbb{C}^{\mathbb{Z}/N\mathbb{Z}}$ *is given by*

$$(\mathcal{F}a)(k)=\frac{1}{\sqrt{N}}\sum_{j=0}^{N-1}a(j)e^{-2\pi i\frac{jk}{N}}.$$

Proof If M is the usual generator matrix for H, then B^nM is the generator matrix of H_n, where

$$B=\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}.$$

Let

$$B^n=\begin{pmatrix}p_n & q_n\\ -q_n & p_n-q_n\end{pmatrix}.$$

By Theorem 3 it suffices to show that p_n and q_n are relatively prime. This is proved by induction. It is clearly true for $n=1$. Assume that p_n and q_n are relatively prime, and by way of contradiction that p_{n+1} and q_{n+1} have a common prime factor $d\neq 1$. The pair (p_n,q_n) satisfies the recurrence

$$(p_{n+1},q_{n+1})=(p_n,q_n)\begin{pmatrix}2 & -1\\ 1 & 3\end{pmatrix}.$$

Solving the above linear system for p_n and q_n we obtain

$$7p_n=3p_{n+1}-q_{n+1},$$
$$7q_n=p_{n+1}+2q_{n+1}.$$

Thus both $7p_n$ and $7q_n$ are divisible by d, which shows that either both p_n and q_n are divisible by d, contradicting that they are relatively prime, or $d=7$. But for no n does 7 divide both p_n and q_n for the following reason. From the Cayley-Hamilton Theorem, $B^2=5B-7\equiv 5B \pmod 7$, from which it readily follows that $B^7=B \pmod 7$. Now we merely check that for the matrices B, B^2, B^3, B^4, B^5, B^6 the entries in the first row are not both congruent to 0 modulo 7. □

Again, to make practical use of Corollary 5, it is helpful to have the bijection $\phi:D_n\to\mathbb{Z}/7^n\mathbb{Z}$ between the standard set D_n of coset representatives for GBT_n and $\mathbb{Z}/7^n\mathbb{Z}$. With respect to complex addition and multiplication, GBT_n is a ring. Corollary 5 shows that GBT_n is an (additive) cyclic group of order 7^n generated by 1. So there is a group isomorphism which readily extends to a ring isomorphism

$$\phi:\mathrm{GBT}_n\to\mathbb{Z}/7^n\mathbb{Z}$$

induced by taking $1\in\mathrm{GBT}_n$ to $1\in\mathbb{Z}/7^n\mathbb{Z}$. This provides the required bijection. To apply ϕ to an arbitrary $\mathbf{d}=\sum_{k=0}^{n-1}d_k\beta^k\in D_n$ use the ring isomorphism:

$$\begin{aligned}\phi(\mathbf{d})&=\sum_{k=0}^{n-1}\phi(d_k)(\phi\beta)^k\\&=\sum_{k=0}^{n-1}[\epsilon_{0k}+\epsilon_{1k}\phi\omega+\epsilon_{2k}(\phi\omega)^2](2-\phi\omega)^k,\end{aligned}$$

where $d_k=\epsilon_{0k}+\epsilon_{1k}\omega+\epsilon_{2k}\omega^2$. So ϕ is completely determined by $\phi\omega$.

The number $\phi\omega$ can be computed recursively with respect to n using only the identities $\omega\equiv 2 \pmod{\beta}$ and $1+\omega+\omega^2=0$. Computing $\phi\omega$ for $n=1,2,3$ should make the general method clear. Since $\omega\equiv 2 \pmod{\beta}$ we have $\phi\omega\equiv 2 \pmod 7$. Therefore

$$\phi\omega=2 \quad \text{if } n=1.$$

(Note that the isomorphism in this case is exactly the same as given in Fig. 2.) Proceeding to the case $n=2$, the above implies that $\phi\omega\equiv 2+7x \pmod{7^2}$ for some integer x. Since $1+\omega+\omega^2=0$ we have $1+\phi\omega+(\phi\omega)^2\equiv 0 \pmod{7^2}$. Therefore

$$0\equiv 1+(2+7x)+(2+7x)^2\equiv 7+35x \pmod{7^2},$$

which implies that $1+5x\equiv 0 \pmod 7$ or $x\equiv 4 \pmod 7$. Therefore

$$\phi\omega=2+7\cdot 4=30 \quad \text{if } n=2.$$

Proceeding to the case $n=3$, since $\phi\omega\equiv 30 \pmod{7^2}$ we have $\phi\omega\equiv 30+7^2x \pmod{7^3}$ for some integer x. Since $1+\omega+\omega^2=0$ we have $1+\phi\omega+(\phi\omega)^2\equiv 0 \pmod{7^3}$.

Therefore

$$\begin{aligned}0 &\equiv 1 + (30 + 7^2x) + (30 + 7^2x)^2\\ &\equiv 931 + 61 \cdot 7^2 x \pmod{7^3},\end{aligned}$$

which implies that $19 + 61x \equiv 0 \pmod 7$ or $x \equiv 6 \pmod 7$. Therefore

$$\phi\omega = 30 + 7^2 \cdot 6 = 324 \quad \text{if } n = 3.$$

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